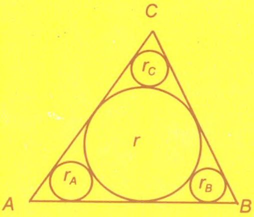
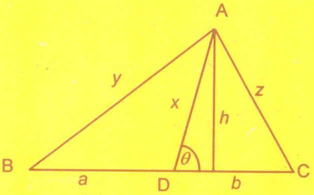
GEMS
from
THE MATHEMATICS TEACHER





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and
G.R.Vijayakumar

Association of Mathematics Teachers of India
1997

GEMS FROM
THE MATHEMATICS TEACHER

Compiled by

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1 Algebra **36**

2 Geometry **63**

3 Trigonometry **105**

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Preface

This volume is a collection of selected problems from various tests conducted by AMTI[1] and NBHM[2]. They are classified into the following groups:

(1) Algebra,

(2) Geometry,

(3) Trigonometry,

(4) Analysis,

(5) Number Theory and

(6) Combinatorics.

Requirement for solving almost all of these problems is exposure to undergraduate level Mathematics. Most of the problems are at the level of Higher Secondary School Mathematics.

The problems being taken from sources of various levels, the effort required to solve them varies a lot; some of the problems are quite easy – most often such problems are included as illustrations of using various problem-solving techniques; in such situations, the reader is expected to find the shortest/simplest solutions. Some problems are quite tough – just at the level of the problems of International Mathematical Olympiads.

For the enthusiastic student who prepares for participating in various competitions such as RMO[3], INMO[4], this book should be a good training-ground. This is one reason for the inclusion of various alternative solutions.

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[1] The Association of Mathematics Teachers of India.

[2] National Board for Higher Mathematics.

[3] Regional Mathematical Olympiad.

[4] Indian National Mathematical Olympiad.

The authors would like to thank Professor M. S. Rangachari for his encouragement and for the enthusiasm shown by him in bringing out this book.

We will appreciate very much receiving suggestions from the readers regarding this volume.

S. Muralidharan and G. R. Vijayakumar

Sources

Here is the complete list of all tests from which the problems have been selected*.

AMTI Junior Level Final Test 1990	JF-1990	**26** (3-4) 1990
AMTI Inter Level Final Test 1990	IF-1990	**26** (3-4) 1990
AMTI Junior Level Screening Test 1991	JS-1991	**27** (1-4) 1991
AMTI Inter Level Screening Test 1991	IS-1991	**27** (1-4) 1991
AMTI Junior Level Final Test 1991	JF-1991	**27** (1-4) 1991
AMTI Inter Level Final Test 1991	IF-1991	**27** (1-4) 1991
AMTI Senior Level Final Test 1991	SF-1991	**27** (1-4) 1991
NBHM INMO 1991	INMO-1991	**27** (1-4) 1991
AMTI Junior Level Screening Test 1992	JS-1992	**29** (1-2) 1993

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* Each source is followed by the abbreviation used for that in this volume and the reference to the issue of 'The Mathematics Teacher' in which the solution-set has been published.

AMTI Inter Level Screening Test 1992
IS-1992 29 (1-2) 1993

AMTI Junior Level Final Test 1992
JF-1992 29 (1-2) 1993

AMTI Inter Level Final Test 1992
IF-1992 29 (1-2) 1993

AMTI Senior Level Final Test 1992
SF-1992 29 (1-2) 1993

AMTI Supplementary Inter Test 1992
SI-1992 29 (1-2) 1993

NBHM RMO 1992
RMO-1992 29 (1-2) 1993

NBHM INMO 1992
INMO-1992 29 (1-2) 1993

AMTI Junior Level Screening Test 1993
JS-1993 30 (1-2) 1994

AMTI Junior Level Final Test 1993
JF-1993 30 (1-2) 1994

AMTI Inter Level Screening Test 1993
IS-1993 30 (1-2) 1994

AMTI Inter Level Final Test 1993
IF-1993 30 (1-2) 1994

AMTI Senior Level Test 1993
S-1993 30 (1-2) 1994

NBHM RMO 1993
RMO-1993 30 (1-2) 1994

NBHM INMO 1993
INMO-1993 **30** (1-2) 1994

AMTI Junior Level Screening Test 1994
JS-1994 **31** (1-2) 1995

AMTI Inter Level Screening Test 1994
IS-1994 **31** (1-2) 1995

AMTI Junior Level Final Test 1994
JF-1994 **31** (1-2) 1995

AMTI Inter Level Final Test 1994
IF-1994 **31** (1-2) 1995

AMTI Senior Level Test 1994
S-1994 **31** (1-2) 1995

NBHM RMO 1994
RMO-1994 **31** (1-2) 1995

NBHM INMO 1994
INMO-1994 **31** (1-2) 1995

NBHM INMO 1995
INMO-1995 **31** (1-2) 1995

Notation

$\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of all natural numbers.

$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of all integers.

$\mathbb{Q} = \{p/q : p \in \mathbb{Z},\ q \in \mathbb{N}\}$, the set of all rational numbers.

$\mathbb{R}$ denotes the set of all real numbers and

$\mathbb{C}$, the set of all complex numbers.

$\mathbb{R}^*$ is the set of all non-zero reals.

For any set X, $P(X)$ is the power set of X; i.e., the set of all subsets of X.

The greatest common divisor of two integers a and b is denoted by $\gcd(a, b)$; their least common multiple is $\operatorname{lcm}(a, b)$. When we write $(a, b) = d$ (resp. $\langle a, b\rangle = d$) where $d \in \mathbb{N}$, we mean that $\gcd(a, b) = d$ (resp. $\operatorname{lcm}(a, b) = d$).

For $a, b \in \mathbb{Z}$, $a \mid b$ means that b is divisible by a; $a \nmid b$ means that a does not divide b.

For any real r, $\lfloor r \rfloor = \max\{k \in \mathbb{Z} : k \leq r\}$ and $\lceil r \rceil = \min\{k \in \mathbb{Z} : k \geq r\}$.

For $m \in \mathbb{N}$ and $n \in \mathbb{Z}$, $\binom{m}{n}$ is the binomial co-efficient that stands for the number of ways of choosing n things out of m things. (Note that $\binom{m}{n} = 0$ when n is negative or $m < n$.)

If n is any positive integer greater than 1, the number of positive integers less than n and prime to n is denoted by $\varphi(n)$; $\varphi(1)$ is defined to be 1.

The conjugate of a complex number z is denoted by $\overline{z}$.

$|z|$ is the absolute value of z.

$\ell \parallel m$ means that the lines ℓ and m are parallel.

$\ell \perp m$ means that ℓ is perpendicular to m.

$\triangle ABC \equiv \triangle PQR$ means that the triangles ABC and PQR are congruent.

$\triangle ABC \cong \triangle PQR$ means that the triangles ABC and PQR are similar.

$\triangle ABC \doteq \triangle PQR$ means that the triangles ABC and PQR are equal in area.

When a triangle's vertices are A, B, C, we use the following convention: a, b, c denote the lengths of the sides BC, CA, AB respectively. $s = \frac{1}{2}(a + b + c)$. R and r are respectively the measures of circumradius and inradius. $\triangle$ is the area of the triangle. S, I, O and G are respectively the circumcenter, incenter, orthocenter and centroid.

Problems

❶

Algebra

1.01 Let G be a finite group with identity e. If g and h are two elements in G such that $g^3 = e$ and $ghg^{-1} = h^2$, find the order of h. S-1994

1.02 If x, y, z are positive reals such that $x^3y^2z^4 = 7$, show that $2x + 5y + 3z \geq 9\left(\frac{525}{2^7}\right)^{1/9}$ IF-1994

1.03 If a, b, c, d are four non-negative reals such that $a + b + c + d = 1$, show that $ab + bc + cd \leq \frac{1}{4}$. INMO-1993

1.04 If a, b, c are distinct real numbers, find the number of real solutions to

$$\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + 1 = 0.$$

IS-1992

1.05 If $0 \leq a < b$ and $k > 0$, show that

$$\sqrt{\sqrt{a+k} - \sqrt{a}} > \sqrt{\sqrt{b+k} - \sqrt{b}}.$$

JF-1992

1.06 If $x^5 - x^3 + x = a$ prove that $x^6 \geq 2a - 1$.

INMO-1994

1.07 If $\dfrac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}} = 3$ find x. JS-1992

1.08 If m and n are the roots of the equation

$$(x + p)(x + q) - k = 0,$$

find the roots of

$$(x-m)(x-n)+k=0.$$ JS-1994

1.09 Find the real roots of the equation

$$x^2+2ax+\frac{1}{16}=-a+\sqrt{a^2+x-\frac{1}{16}}$$

where $0<a<\frac{1}{4}$. IF-1994

1.10 Solve:

$$\sqrt{x+14-8\sqrt{x-2}}+\sqrt{x+23-10\sqrt{x-2}}=3.$$

SI-1992

1.11 Find out the area of the triangle whose sides are 8, 17, 15. JS-1994

1.12 If $a=(\sqrt{3}+\sqrt{2})^{-3}$ and $b=(\sqrt{3}-\sqrt{2})^{-3}$, find the value of $(a+1)^{-1}+(b+1)^{-1}$ JS-1994

1.13 If a, b, c are positive integers such that

$$\frac{a}{3}=\frac{b}{4}=\frac{c}{5} \quad \text{and} \quad abc=1620,$$

find the value of b. JS-1994

1.14 If a, b, c are positive real numbers such that

$$\frac{a}{2b-c}=\frac{2b}{3a+c}=\frac{a}{b},$$

determine the common value of these fractions. JS-1994

1.15 Show that if n is congruent to 2 or 3 modulo 4, then it is not possible to get a rearrangement $(x_1, x_2, \ldots, x_n)$ of

$(1, 2, \ldots, n)$ such that $|x_1 - 1|, |x_2 - 2|, \ldots, |x_n - n|$ are all distinct. IF-1994

1.16 ABC is a right angled triangle with $\angle B = 90^\circ$. M is the midpoint of AC and $BM = \sqrt{117}$. Sum of the lengths of the other two sides AB and BC is 30. Find the area of the triangle. IS-1992

1.17 If $|x - 1| = 2x$, find the value of x. IS-1992

1.18 Find the real numbers x for which

$$|x + 1| + 2\,|x - 2| < 6.$$

JS-1992

1.19 If $x = a + b$, $y = a\omega + b\omega^2$ and $z = a\omega^2 + b\omega$, where ω is a non-real cube root of unity, find $x^3 + y^3 + z^3$ in terms of a and b. IS-1994

1.20 Find the real numbers x such that $x + 3$, $2x + 5$ and $3x + 2$ are the sides of a right-angled triangle. IS-1994

1.21 Show that the polynomials in the set

$$\mathcal{P} = \{P_k : P_k(x) = x^{5k+4} + x^3 + x^2 + x + 1,\ k \in \mathbb{N}\}$$

have a common non-trivial polynomial divisor. IF-1993

1.22 If f is a real valued function defined on $[0, 1]$ such that $f(0) = f(1) = 0$ and $|f(x_1) - f(x_2)| < |x_1 - x_2|$ for all distinct $x_1, x_2 \in [0, 1]$, prove that $|f(x_1) - f(x_2)| < \frac{1}{2}$ for all $x_1, x_2 \in [0, 1]$. S-1993

1.23 In base R_1, two fractions F_1 and F_2 are expanded as $0.3737\cdots$ and $0.7373\cdots$ respectively; in another base R_2, they are $0.2525\cdots$ and $0.5252\cdots$. Find the sum of R_1 and R_2 written in base 10. IS-1993

1.24 Find all pairs (x, y) of real numbers that satisfy the equation

$$\left(\sin^2 x + \frac{1}{\sin^2 x}\right)^2 + \left(\cos^2 x + \frac{1}{\cos^2 x}\right)^2 = 12 + \frac{1}{2}\sin y.$$

SF-1992

1.25 Let $0 < a, b, c, d, e < 1$. Show that

$$1 - a - b - c - d - e < (1-a)(1-b)(1-c)(1-d)(1-e).$$

JF-1990

1.26 Prove that there is an uncountable chain in $(P(\mathbb{N}), \subset)$ where $\subset$ is the set-inclusion relation. (A subset $\mathcal{F}$ of $P(\mathbb{N})$ is called a *chain* in $P(\mathbb{N})$, if for all $A, B \in \mathcal{F}$, either $A \subset B$ or $B \subset A$.) S-1993

1.27 The roots of a polynomial $x^3 + ax^2 + bx + c$ are of the form u, v, uv. If a, b, c are rationals and $a \neq 1$, show that uv is also rational. Can we relax the condition that $a \neq 1$? S-1993

1.28 If a, b, c are the sides of a triangle and $a + b + c = 2$, prove that $a^2 + b^2 + c^2 + 2abc < 2$. IF-1993

1.29 If $g(x)$ and $h(x)$ are polynomials with real coefficients and $f(x) = g(x^3) + xh(x^3)$ is divisible by $x^2 + x + 1$, show that $g(x)$ and $h(x)$ are both divisible by $x - 1$. IS-1994

1.30 If a, b, c, d are four positive real numbers such that $abcd = 1$, prove that $(1+a)(1+b)(1+c)(1+d) \geq 16$.

RMO-1993

1.31 Two brothers had a flock of sheep. They sold the flock and got as many rupees for every sheep as was the number of the sheep in the flock. The money was shared in the following way: First, the elder brother took ten rupees from the cash, then the younger brother took ten rupees, after which the elder brother took ten rupees again, and so on. Finally, it turned out that at the last stage when it was the younger brother's turn to take money there remained less than ten rupees. Therefore the younger brother took the rest of the money and the elder brother gave him his knife for the sharing to be fair. Find the cost of the knife.

JF-1994

1.32 A balance has arms of unequal length and pans of unequal weight. A certain article appears to weigh Q_1 or Q_2 according as it is put into one scale pan or the other. Similarly another article appears to weigh R_1 or R_2 in that order. Show that the true weight of the article which appears to weigh the same in whichever scale pan it is put is

$$\frac{Q_1R_2 - Q_2R_1}{Q_1 - Q_2 - R_1 + R_2}$$

IF-1990

1.33 If $f(x) = x^3 + ax + b$ is divisible by $(x-1)^2$, find the remainder obtained when $f(x)$ is divided by $x+2$.

JS-1994

1.34 If $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}$, then prove that

$$\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} = \frac{1}{a^n + b^n + c^n}$$

for any odd integer n. IF-1990

1.35 Find all real numbers x, y satisfying $\log_3 x + \log_2 y = 2$ and $3^x - 2^y = 23$. IF-1992

1.36 A positive integer > 1 is *highly divisible* if it has more divisors than all the previous positive integers. Show that any highly divisible number is even. IF-1990

1.37 An ordered triplet (a, b, c) where a, b, c are non-zero reals is said to be *good* if each of a, b, c is the product of the other two. Find all such triplets. JS-1993

1.38 If a, b are positive real numbers such that $a^2+b^2 = 1$, show that $a + b + \dfrac{1}{ab} \geq 2 + \sqrt{2}$. SF-1991

1.39 If r and s are the roots of $ax^2 + bx + c = 0$, find the equation whose roots are $ar + b$ and $as + b$. JS-1991

1.40 If all the roots of $x^4 - 8x^3 + 24x^2 + bx + c = 0$ are positive reals, find the value of b. JS-1991

1.41 Find the fixed points of the function $f : \mathbb{R} \mapsto \mathbb{R}$ which satisfies $f(f(x)) - f(x) = ax + b$ for all $x \in \mathbb{R}$, where $a \neq 0$. SF-1991

1.42 If $xy : yz : zx = 2 : 3 : 4$, find $\dfrac{y}{zx} : \dfrac{z}{xy}$. JS-1991

1.43 Find the real points (x, y) satisfying

$$3x^2 + 3y^2 - 4xy + 10x - 10y + 10 = 0.$$ IS-1991

1.44 If a, b, c are real such that $a+b+c > 0$, $ab+bc+ca > 0$ and $abc > 0$, show that a, b, c are positive. JF-1991

1.45 One root of the equation $x^4 - 5x^3 + ax^2 + bx + c = 0$ where a, b, c are rational, is $3 + \sqrt{2}$. If all other roots are real, find the largest value of c. IS-1991

1.46 Let a, b, c be real numbers with $0 < a < 1$, $0 < b < 1$, $0 < c < 1$ and $a + b + c = 2$. Show that

$$\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8.$$ INMO-1991

1.47 If $a, b > 0$ and $a + b = 1$, show that

$$\sqrt[3]{11 + \frac{1}{a^4}} + \sqrt[3]{11 + \frac{1}{b^4}} \geq 6.$$ IF-1991

1.48 Show that there exist no rational numbers a, b, c, d such that $(a+b\sqrt{2})^{100} + (c+d\sqrt{2})^{100} = 7+5\sqrt{2}$. IF-1993

1.49 Solve for x, y and z:

$$\begin{aligned} xy + x + y &= 23 \\ yz + y + z &= 31 \\ zx + z + x &= 47. \end{aligned}$$ IF-1990

1.50 Let $p(x) = x^2 + ax + b$ be a quadratic polynomial in which a and b are integers. Given any integer n, show that there is an integer m such that $p(n)p(n+1) = p(m)$.

INMO-1993

1.51 For any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, define

$$x^{(0)} = 1 \quad \text{and} \quad x^{(n)} = x(x-1)\cdots(x-n+1).$$

Show that

$$(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}$$ IF-1990

1.52 f is a polynomial with integer coefficients such that there exist four distinct integers a_1, a_2, a_3, a_4 with $f(a_1) = f(a_2) = f(a_3) = f(a_4) = 1991$. Show that there exists no integer a such that $f(a) = 1993$. S-1993

1.53 Find all values of a, b for which the system of equations

$$xyz + z = a, \quad xyz^2 + z = b, \quad x^2 + y^2 + z^2 = 4$$

has only one real solution. IF-1992

1.54 If $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying for all $x \in \mathbb{R}$

$$\begin{aligned} f(-x) &= -f(x), \\ f(x+1) &= f(x) + 1 \quad \text{and} \\ f\left(\frac{1}{x}\right) &= \frac{f(x)}{x^2} \quad (\text{when } x \neq 0), \end{aligned}$$

prove that $f(x) = x$ for all real values of x. INMO-1994

1.55 Solve:

$$\begin{aligned} (1+x)(1+y) &= 4y \\ (1+y)(1+z) &= 4z \\ (1+z)(1+x) &= 4x. \end{aligned}$$

JF-1994

1.56 The roots x_1, x_2, x_3 of the equation $x^3 + ax + a = 0$ where a is a non-zero real, satisfy $\dfrac{x_1^2}{x_2} + \dfrac{x_2^2}{x_3} + \dfrac{x_3^2}{x_1} = -8$. Find x_1, x_2, x_3. IF-1994

1.57 If x, y, z are three real numbers such that $x+y+z = 4$ and $x^2 + y^2 + z^2 = 6$, then show that each of x, y, z lies in the closed interval $[2/3, 2]$. Can x attain the extreme values $2/3$ and 2? INMO-1992

1.58 Find all the solutions of the system of equations: $y = 4x^3 - 3x$, $z = 4y^3 - 3y$ and $x = 4z^3 - 3z$.

IF-1993

1.59 Let $(u_n)_{n=1}^{\infty}$ be a sequence of positive reals such that $u_4 = 4$ and for all $n \in \mathbb{N}$,

$$\frac{1}{u_1u_2} + \frac{1}{u_2u_3} + \cdots + \frac{1}{u_nu_{n+1}} = \frac{u_n}{u_{n+1}}.$$

Prove that $u_n = n$ for all $n \in \mathbb{N}$. SF-1991

1.60 Find all values of a for which the system of equations

$$2^{|x|} + |x| = y + x^2 + a \qquad x^2 + y^2 = 1$$

has only one solution. (a, x, y are real numbers.)

SF-1992

1.61 Find all real x, y satisfying $x^3 + y^3 = 7$ and $x^2 + y^2 + x + y + xy = 4$. JF-1991

1.62 Find all the real numbers x, y satisfying $x^8 + y^8 = 8xy - 6$. IF-1992

1.63 Let $a_1, a_2, a_3, \ldots, a_n$ be real numbers greater than 1 such that $|a_k - a_{k+1}| < 1$ for $1 \leq k \leq n-1$. Show that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} < 2n - 1.$$ INMO-1995

Geometry

2.01 If $\angle A$ of a triangle ABC is doubled and the lengths of the sides AB and AC are kept the same, the area of the triangle remains the same. Find $\angle A$. IS-1994

2.02 Show that the centroid of a triangle coincides with that of the triangle formed by joining the mid-points of its sides. JF-1990

2.03 Suppose PQR is an equilateral triangle inscribed in a triangle ABC such that P lies on BC, Q on CA and R on AB. If

$$\frac{BP}{PC}=\frac{CQ}{QA}=\frac{AR}{RB}$$

prove that $\triangle ABC$ is also equilateral. IF-1993

2.04 In an acute-angled triangle ABC, $\angle A = 30^\circ$, O is the orthocenter and M is the mid point of BC. On the line OM, T is the point such that $OM = MT$. Show that $AT = 2BC$. INMO-1995

2.05 Show that any circle with center $(\sqrt{2}, \sqrt{3})$ cannot pass through more than one lattice point. (Lattice points are points with both coordinates integral.) IF-1990

2.06 In a triangle ABC, angle A is twice angle B. Show that $a^2 = b(b+c)$. INMO-1992

2.07 A cyclic octagon $ABCDEFGH$ has sides a, a, a, a, b, b, b, b respectively. Find the radius of the circumcircle of this octagon. RMO-1992

2.08 A circle passes through the vertex C of a rectangle $ABCD$ and touches its sides AB and AD at M and N respectively. If the distance from C to the line segment MN is equal to 5 units find the area of the rectangle $ABCD$.

INMO-1994

2.09 AD is the internal bisector of angle A in a triangle ABC. Show that the line through D drawn parallel to the tangent of the circumcircle at A touches the inscribed circle.

SI-1992

2.10 Given are two concentric circles of radii R, r. From a point P on the smaller circle, a straight line is drawn to intersect the larger circle at B and C. The perpendicular to BC at P intersects the smaller circle at A. Show that

$$PA^2 + PB^2 + PC^2 = 2(R^2 + r^2).$$

SI-1992

2.11 An ellipse has center O and the ratio of the lengths of the axes is $2 + \sqrt{3}$. If P is a point on the ellipse, prove that the acute angle between the tangent of the ellipse at P and PO is at least $\pi/6$.

S-1993

2.12 S is the circumcenter of a $\triangle ABC$ and M is the mid-point of the median through A. Join SM and produce it to N so that $SM = MN$ Show that N lies on the altitude through A.

IF-1992

2.13 A circle of radius r touches a straight line at a point M. Two points A and B are chosen on this line on opposite sides of M such that $MA = MB = a$. Find the radius of the circle passing through A and B and touching the given circle. IS-1992

2.14 A rhombus has half the area of the square with the same side-length. Find the ratio of the long diagonal to the short one. IS-1992

2.15 AB is a diameter of a circle of radius 10. C is a point on the circle such that arc $BC = \frac{5\pi}{3}$. The bisector of $\angle ACB$ cuts the circle at D. Find the length of CD.

IS-1992

2.16 Let $ABCD$ be a rectangle with $AB = a$ and $BC = b$. Suppose r_1 is the radius of the circle passing through A and B and touching CD; r_2 is the radius of the circle passing through B and C and touching AD. Show that

$$r_1 + r_2 \geq \frac{5}{8}(a + b).$$

RMO-1993

2.17 A quadrilateral $ABCD$ is inscribed in a circle with AD as diameter. If $AD = 4$ and $AB = BC = 1$, find the length of CD. IS-1993

2.18 The diagonals AC, BD of a quadrilateral $ABCD$ intersect at an interior point O. The areas of the triangles AOB and COD are s_1 and s_2 respectively and the area of the quadrilateral is s. Prove that $\sqrt{s_1} + \sqrt{s_2} \leq \sqrt{s}$. Also prove that the equality holds if and only if the lines AB and CD are parallel. IF-1993

2.19 Let ABC be an acute angled triangle and CD be the altitude through C. If $AB = 8$ and $CD = 6$, find the distance between the mid-points of AD and BC.

RMO-1993

2.20 Let P be a point on the curve $xy = 1$; let Q be the foot of the perpendicular from O the origin, on the tangent at P. Find the area in the first quadrant enclosed by the locus of Q. S-1993

2.21 Prove that if one angle of a triangle is equal to 120°, then the triangle formed by the feet of the angle bisectors is right angled. IF-1992

2.22 Let AC and BD be two chords of a circle with center O such that they intersect at right-angles inside the circle at the point M. Suppose K and L are the mid-points of the chords AB and CD respectively. Prove that $OKML$ is a parallelogram. RMO-1994

2.23 The diagonals AC and BD of a cyclic quadrilateral $ABCD$ intersect at P. Let O be the circumcenter of the triangle APB and H be the orthocenter of the triangle CPD. Show that the points H, P, O are collinear. INMO-1993

2.24 Let ABC be an equilateral triangle with side 10 and let P be a point inside the triangle. If PD, PE and PF are the perpendiculars to the three sides, find out whether the sum $PD + PE + PF$ depends on the position of P.

IS-1993

2.25 Let M be the midpoint of the side AB of a triangle ABC. Let P be a point on AB between A and M and let MD be drawn parallel to PC intersecting BC at D. Find out whether the ratio of the area of triangle BPD to that of ABC depends upon the position of P. IS-1993

2.26 $ABCDE$ is a convex pentagon inscribed in a circle of radius 1 with AE as a diameter. If $AB = a$, $BC = b$, $CD = c$ and $DE = d$, prove that

$$a^2 + b^2 + c^2 + d^2 + abc + bcd < 4.$$ SF-1992

2.27 Given a triangle ABC, how will you find points P, Q, R on the sides AB, BC, CA such that $APQR$ is a rhombus? Show that the area of this rhombus can not exceed half of the area of the triangle ABC. IF-1994

2.28 AB, AC are two radii of a circle inclined at an angle of 60°. Upon AC a point P is taken such that a circle can be described with center P to touch the first circle internally and also to touch the circle with AB as diameter, externally. Prove that $AP = \frac{4}{5}AC$. JF-1990

2.29 ABC is a triangle right angled at A and two circles with radii r_1 and r_2 respectively touch both AB and AC; one of them touches the circumcircle of ABC internally and the other, externally. Show that $4\triangle ABC = r_1 r_2$.

INMO-1993

2.30 Two parallel straight lines intersect the x-axis cutting off a segment AB of length 4; they cut off a segment CD of length 3 on the y-axis. Find the perpendicular distance between these lines. IS-1992

2.31 M is an interior point of a triangle ABC. Bisectors of interior angles BMC, CMA, AMB intersect BC, CA,

AB respectively at X, Y, Z. Show that AX, BY, CZ are concurrent; if P is the point of concurrence and

$$\frac{PA}{PX}\cdot\frac{PB}{PY}\cdot\frac{PC}{PZ}=8$$

show that M is the circumcenter of $\triangle ABC$. IF-1991

2.32 If $A_1, A_2, \ldots, A_n$ is an n-sided regular polygon such that

$$\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4},$$

find n, the number of sides of the polygon. INMO-1992

2.33 In a $\triangle ABC$, AD is the bisector of the angle A. If $AB=3$, $AC=6$ and $BC=3\sqrt{3}$, find the length of AD.

IS-1991

2.34 $ABCD$ is a convex quadrilateral with $AB=3$, $BC=5$ and $CD=7$. If the bisectors of the four internal angles are concurrent, find the value of $(AB+BC+CD+DA)/DA$.

IS-1992

2.35 $ABCD$ is a parallelogram and a transversal cuts AB, AD, AC at P, Q, R respectively. Prove that

$$\frac{AC}{AR}=\frac{AB}{AP}+\frac{AD}{AQ}.$$

IF-1991

2.36 $ABCD$ is a quadrilateral and P, Q are the midpoints of AB, CD respectively. If AQ, DP intersect at X and BQ, CP at Y, prove that the area of the quadrilateral $PXQY$ is equal to $\triangle AXD+\triangle BYC$. RMO-1992

2.37 If S is the circumcenter of a $\triangle ABC$, AS meets BC at M, BS meets CA at N and CS meets AB at P, prove that

$$\frac{1}{AM} + \frac{1}{BN} + \frac{1}{CP} = \frac{2}{R}$$

where R is the circumradius. IF-1991

2.38 Let G be the centroid of the triangle ABC in which the angle at C is obtuse and let AD and CF be medians from A and C respectively onto the sides BC and AB. If the four points B, D, G and F are concyclic, show that

$$\frac{AC}{BC} > \sqrt{2}.$$

If, further, P is a point on the line BG extended such that $AGCP$ is a parallelogram, show that the triangles ABC and GAP are similar. INMO-1994

2.39 $ABCD$ is a cyclic quadrilateral with $AC \perp BD$. If AC and BD intersect at M and R is the radius of the circumcircle, prove that $MA^2 + MB^2 + MC^2 + MD^2 = 4R^2$.
RMO-1992

2.40 Points S and T trisect the hypotenuse PQ of the right angled $\triangle PQR$. If $RS = 7$ and $RT = 9$, find the length of ST. IS-1991

2.41 In a triangle ABC, $AB = AC$. A transversal intersects AB and AC internally at K and L respectively; it intersects BC produced at M. If $KL = 2LM$ find $\frac{KB}{LC}$.
JS-1991

2.42 In a quadrilateral $ABCD$, a circle with center at the mid-point of AB touches the sides BC, CD and AD. Show that $AB^2 = 4 \cdot AD \cdot BC$. SI-1992

2.43 ABC is a triangle and D and E are interior points of the sides AB and BC respectively such that $\dfrac{AD}{DB} = \dfrac{1}{3}$ and $\dfrac{CE}{EB} = 3$. If AE and CD intersect at F find $\dfrac{CF}{FD}$.
IS-1991

2.44 In a trapezium $ABCD$, $AB \parallel CD$; the measure of angle D is twice that of B. If $AD = a$ and $CD = b$, find the length of AB.
IS-1993

2.45 Constructed externally on the sides AB, AC of a $\triangle ABC$ are equilateral triangles ABX and ACY. If P, Q, R are the mid-points of AX, AY, BC respectively, show that $\triangle PQR$ is equilateral.
JF-1991

2.46 In a triangle ABC, $BC \neq CA$. The bisectors of $\angle A$ and $\angle B$ meet BC and CA at X and Y respectively; the two bisectors intersect at I. If $IX = IY$, prove that $\angle C = 60^\circ$.
JF-1993

2.47 Show that any rectangle inscribed in an ellipse has its sides parallel to the axes of the ellipse.
S-1994

2.48 In a triangle ABC, $AB = AC$, $\angle A = 80^\circ$ and S is the circumcenter. Bisectors of angles ACS and ABS meet BS and CS respectively at X and Y. Find the angles of $\triangle AXY$.
JF-1992

2.49 In a $\triangle ABC$, M is the mid-point of BC, P is any point on AM and PE, PF are perpendiculars to AB, AC

respectively. If EF is parallel to BC, show that $\angle A = 90^\circ$ or $\angle B = \angle C$. JF-1992

2.50 If P is a point inside a rectangle $ABCD$ such that $PA = 3$, $PB = 4$ and $PC = 5$ find PD. JF-1994

2.51 In $\triangle ABC$, the median from A is perpendicular to the median from B. If $BC = 7$ and $AC = 6$ find AB.

JS-1991

2.52 $ABCD$ is a square such that A lies on the positive y-axis and B lies on the positive x-axis. If D is the point (12,17) find the coordinates of C. IF-1994

2.53 Given any acute angled $\triangle ABC$, let points X, Y, Z be located as follows: X is the point where the altitude from A on BC meets the outward-facing semicircle drawn on BC as diameter. Points Y and Z are located similarly. Prove that

$$(\triangle BCX)^2 + (\triangle CAY)^2 + (\triangle ABZ)^2 = (\triangle ABC)^2.$$

INMO-1991

2.54 O is the orthocenter of a triangle ABC and K, L, M are its mirror images on the three sides. Show that the triangle KLM has the same circumcenter as triangle ABC.

IF-1990

❸

Trigonometry

3.01 In a triangle ABC points P, Q, R are chosen on BC, CA, AB respectively. If S_1, S_2, S_3 and S are the areas of the triangles AQR, BRP, CPQ and PQR respectively, prove that

$$S\left(\frac{1}{S_1}+\frac{1}{S_2}+\frac{1}{S_3}\right) \geq 3.$$ IF-1993

3.02 Find all positive integers n such that

$$\sin\frac{\pi}{2n}+\cos\frac{\pi}{2n}=\frac{\sqrt{n}}{2}.$$ IF-1992

3.03 Let a, b, c be the lengths of the sides of a triangle and r its inradius; show that

$$r < \frac{a^2+b^2+c^2}{3(a+b+c)}.$$ JF-1991

3.04 In any triangle ABC, show that

$$\sum_{A,B,C}\frac{\sqrt{\sin A}}{\sqrt{\sin B}+\sqrt{\sin C}-\sqrt{\sin A}} \geq 3.$$ IF-1993

3.05 In a triangle ABC, the incircle touches the sides BC, CA and AB respectively at D, E and F. If the radius of the incircle is 4 units and if BD, CE and AF are consecutive integers, find the sides of the triangle ABC.

RMO-1994

3.06 Find all triangles such that the lengths of the sides are integers and the radius of the inscribed circle is 1.

SI-1992

3.07 Given a $\triangle ABC$, define the quantities x, y, z as follows:

$$x = \tan\frac{B-C}{2}\tan\frac{A}{2},$$
$$y = \tan\frac{C-A}{2}\tan\frac{B}{2}$$
$$\text{and}\quad z = \tan\frac{A-B}{2}\tan\frac{C}{2}.$$

Prove that $x + y + z + xyz = 0$. INMO-1991

3.08 If $\cos^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{y}{b}\right) = \alpha$, prove that

$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha.$$ IF-1991

3.09 If a, b, c are the lengths of the sides of $\triangle ABC$, show that there exist positive real numbers x, y, z such that $a = y + z$, $b = z + x$, $c = x + y$. Express the inradius r and the circumradius R in terms of x, y, z. Hence deduce that

$$\frac{R}{r} \geq \frac{b}{c} + \frac{c}{b}.$$ IF-1991

3.10 If in an acute angled triangle ABC

$$8(\cos^3 A + \cos^3 B + \cos^3 C - 3\cos A\cos B\cos C) +$$
$$36(\cos A\cos B + \cos B\cos C + \cos C\cos A) = 27,$$

show that the triangle is equilateral. IF-1991

3.11 Let ABC be a triangle and a circle Γ_1 be drawn lying inside the triangle, touching its incircle Γ externally and also touching the two sides AB and AC. Show that the ratio of the radii of the circles Γ_1 and Γ is equal to $\tan^2\left(\frac{\pi - A}{4}\right)$.

INMO-1995

3.12 Show that in any triangle ABC,

$$\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}$$

with equality holding if and only if the triangle is equilateral.

SF-1992

3.13 Show that in a triangle ABC, the incircle passes through the circumcenter if and only if

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{\sqrt{2}-1}{4}$$ SF-1992.

3.14 In a triangle ABC, $(a+b+c)abc = 16\Delta^2$. Show that the triangle is equilateral. SI-1992

3.15 Find all values of x which satisfy

$$\tan x = \tan(x+10^\circ)\tan(x+20^\circ)\tan(x+30^\circ).$$ SF-1992

3.16 ABC is a triangle with $\angle A = 30^\circ$, $\angle B = 60^\circ$ and $AB = 10$. Find the length of the shorter trisector of $\angle C$.

JS-1993

3.17 Show that in any triangle ABC

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} - \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$$
$$\leq \frac{8}{27}\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$ IF-1992

4

Analysis

4.01 Let f be a continuous function on $\mathbb{R}$ such that for all $n \in \mathbb{N}$, $f(n) = 0$ and for all $a, b \in \mathbb{R}$,

$$f(a) = 0 = f(b) \Rightarrow f\left(\frac{a+b}{2}\right) = 0.$$

Show that $f(x) = 0$ for all real x. S-1993

4.02 Determine all functions $f : \mathbb{R} - \{0, 1\} \mapsto \mathbb{R}$ satisfying the functional relation

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}.$$

INMO-1992

4.03 Find all continuous functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f(x) - 2f\left(\frac{x}{2}\right) + f\left(\frac{x}{4}\right) = x^2.$$

SF-1992

4.04 Define a sequence $(x_n)_{n=1}^{\infty}$ by

$$x_1 = \sqrt{2}; \quad x_{n+1} = \sqrt{2}^{x_n}, \; n \in \mathbb{N}.$$

Does this sequence converge? If so, find the limit.

IS-1993

4.05 Find all continuous functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f(x) - f\left(\frac{x}{3}\right) = \frac{2x}{3} \quad \forall\, x \in \mathbb{R}.$$

IF-1991

4.06 Show that

$$\lim_{n \to \infty} \frac{1}{n^{n-1}} \left(\sum_{k=0}^{n} \binom{n}{k} \frac{n^k}{k+1} \right) = e.$$

S-1994

4.07 An ant crawls at the rate of 10 cm per minute along a rubber band which can be stretched uniformly. Suppose that the rubber band is initially one meter long and it is stretched an additional meter at the end of each minute. If the ant begins at one end of the band, does it reach the other end? Why? S-1994

4.08 If $f : \mathbb{R}^* \mapsto \mathbb{R}$ is a continuous function such that $f(xy) = f(x) + f(y)\ \forall\ x, y \in \mathbb{R}^*$ and $f(\alpha) = 0$ for some α with $|\alpha| \neq 1$, show that $f \equiv 0$. SF-1991

❺

Number Theory

5.01 If $\frac{37}{13} = 2 + \cfrac{1}{x + \cfrac{1}{y + \cfrac{1}{z}}}$ where x, y, z are positive integers find x, y, z. JS-1992

5.02 Let $f(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$ be a polynomial with integer coefficients. If $f(r/s) = 0$ where r and s are integers prime to one another, show that r divides a_0 and s divides a_n. Hence find all the rational roots of

$$1 + x + x^{100} - x^{500} + x^{1000} = 0.$$

JF-1990

5.03 Let A be a set of 16 positive integers, no two of them have a common divisor. Show that A has two elements whose product is greater than 2000. RMO-1994

5.04 Determine the largest 3-digit prime factor of $\binom{2000}{1000}$

RMO-1992

5.05 Find all integral solutions of

$$x^3 + 5y^3 + 25z^3 - 15xyz = 0.$$

JF-1990

5.06 a, b, c are integers such that $\gcd(a, b) = \gcd(a, b, c)$. Show that the number of pairs of integers (x, y) such that $ax + by = c$ is infinite. JS-1992

5.07 Find all positive integers x, y satisfying

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{20}}.$$

SI-1992

5.08 Find the last digit of $1^5 + 2^5 + \cdots + 99^5$. IS-1992

5.09 Find the number of solutions in positive integers of the equation $3x + 5y = 1008$. JS-1992

5.10 Show that $n^2 - 3n - 19$ is not a multiple of 289 for any integer n. IF-1992

5.11 Let a, b, c, d be four integers. Prove that the product of the six differences

$$a - b,\ a - c,\ a - d,\ b - c,\ b - d,\ c - d$$

is divisible by 12. S-1994

5.12 Prove that for every $n \in \mathbb{N}$ there exists an $m \in \mathbb{N}$ such that n divides every term of the infinite sequence $m + 1,\ m^m + 1,\ m^{m^m} + 1, \ldots$. IF-1993

5.13 In a group of ten persons, each person is asked to write the sum of the ages of all the other nine persons. If all the ten sums form the nine-element set $\{82, 83, 84, 85, 87, 89, 90, 91, 92\}$, find the individual ages of the persons, assuming them to be whole numbers (of years). RMO-1993

5.14 Find a finite sequence of 16 numbers such that

(a) it reads the same from left to right as from right to left,

(b) the sum of any 7 consecutive terms is -1 and

(c) the sum of any 11 consecutive terms is $+1$. S-1994

5.15 Find all triplets of integers in arithmetic progression, the sum of whose squares is 1994. IF-1994

5.16 Prove that there do not exist positive integers x, y, z satisfying $2xz = y^2$ and $x + z = 1994$. IF-1994

5.17 Find out the number of positive integers less than 175 and relatively prime to both 5 and 7. JS-1994

5.18 If n is an integer such that 3 divides $n^2 + n + 1$, find the remainder obtained when n is divided by 3. IS-1994

5.19 Determine all integers n such that $n^4 - n^2 + 64$ is the square of an integer. IF-1994

5.20 If $11 + 11\sqrt{11a^2 + 1}$ is an odd integer where a is a rational number, prove that a is a perfect square.

S-1994

5.21 Find all primes p for which the quotient $(2^{p-1} - 1)/p$ is a square. INMO-1995

5.22 Find all the 6-digit decimal numbers $a_1a_2a_3a_4a_5a_6$ which can be formed by using the digits $1, 2, 3, 4, 5, 6$ once each such that the number $a_1 \cdots a_k$ is divisible by k, for $1 \leq k \leq 6$. RMO-1994

5.23 Show that there is a positive integer n such that $n!$ when written in decimal notation ends with exactly 1993 zeros. INMO-1993

5.24 Find the largest possible value of k for which 3^{11} can be expressed as a sum of k consecutive positive integers.

IS-1993

5.25 Let a, b, c, d, e be consecutive positive integers such that $b + c + d$ is a perfect square and $a + b + c + d + e$ is a perfect cube. What is the smallest possible value of c?

IF-1993

5.26 Find all the sets of consecutive integers whose sum is 1000. JF-1990

5.27 A man born in the first half of the nineteenth century was x years old in the year x^2. When was he born?

JS-1993

5.28 If $(abcd) = (abc) \times (bd)$ where all the three terms are decimal numbers, find the value of b. JS-1994

5.29 Determine all pairs of positive integers (m, n) for which $2^m + 3^n$ is a perfect square. INMO-1992

5.30 Determine the set of all positive integers n for which 3^{n+1} divides $2^{3^n}+1$. Prove that 3^{n+2} does not divide $2^{3^n}+1$ for any positive integer n. INMO-1991

5.31 Show that if the sum of the squares of two whole numbers is divisible by 3,, then each of them is divisible by 3.

JF-1991

5.32 Find the number of positive integers $n \leq 1991$ such that 6 is a factor of $n^2 + 3n + 2$. INMO-1991

5.33 Show that $1^{1989} + 2^{1989} + \cdots + 1990^{1989}$ is a multiple of 1991. JF-1990

5.34 Prove that the ten's digit of any power of 3 is even.

RMO-1993

5.35 Show that $19^{93} - 13^{99}$ is a positive integer divisible by 162. RMO-1993

5.36 Determine the sum of all the divisors d of $19^{88} - 1$ which are of the form $d = 2^a \cdot 3^b$ with $a, b > 0$. S-1993

5.37 Show that there are infinitely many pairs (a, b) of relatively prime integers (not necessarily positive) such that both quadratic equations

$$x^2 + 2ax + b = 0 \quad \text{and} \quad x^2 + ax + b = 0$$

have integral roots. INMO-1995

5.38 If x and y are positive integers such that $1994 = (x + y)^2 + 3x + y$, find y. JS-1994

5.39 A leaf is torn from a paperback novel. The sum of the numbers on the remaining pages is $15,000$. What are the page numbers on the torn leaf? RMO-1994

5.40 Find the remainder when 19^{92} is divided by 92.
INMO-1992

5.41 When $3^{1994} + 2$ is divided by 11 find the remainder.
JS-1994

5.42 Prove that $\sum_{n=14}^{100} n!$ is divisible by 1001 but not by 1000. JF-1993

5.43 Find the last three digits of $25^{63} \times 63^{25}$ IS-1994

5.44 A sequence of numbers $a_0, a_1, a_2 \ldots$ is defined as follows: $a_0 = 0$; for $n = 0, 1, 2, \ldots, a_{n+1} = 3a_n + 1$. Show that a_{1995} is divisible by 11 but a_{1994} is not. JF-1994

5.45 Show that if the difference of two consecutive cubes is a square, then it is the square of the sum of two successive squares. JF-1994

5.46 In any set of 181 square integers, prove that one can always find a subset of 19 numbers, sum of whose elements is divisible by 19. INMO-1994

5.47 Determine the set of integers n for which $n^2+19n+92$ is a square. RMO-1992

5.48 Find the last digit of 2137^{753} JS-1991

5.49 Show that the equation $3x^{10} - y^{10} = 1991$ has no integral solutions. JF-1991

5.50 Find n if $2^{200} - 2^{192} \cdot 31 + 2^n$ is a perfect square. JF-1991

5.51 Find the last two digits of $(56789)^{41}$. IS-1991

5.52 Find all integral solutions of

$$x^4 + y^4 + z^4 - t^4 = 1991.$$

IF-1991

5.53 Let A be any set of 19 distinct integers chosen from the arithmetic progression $1, 4, 7, 10, \ldots, 100$. Show that A must contain at least two distinct numbers whose sum is 104. Find a set of 18 distinct integers from the same progression such that the sum of no two distinct integers from that set equals 104. SI-1992

5.54 For $n \in \mathbb{N}$ let $s(n)$ denote the number of ordered pairs (x, y) of positive integers for which

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}.$$

Determine the set of positive integers n for which $s(n) = 5$.

INMO-1991

5.55 Show that the sum of any two consecutive odd primes is the product of at least three primes which need not be distinct.

JF-1993

5.56 If p is an odd prime and $a, b, c \in \mathbb{Z}$, then show that $p \mid abc(a^n - b^n)(b^n - c^n)(c^n - a^n)$ where $n = \dfrac{p-1}{2}$.

SI-1992

5.57 If n is a positive integer such that

$$\frac{n}{810} = 0.d25d25d25\cdots$$

where d is a single digit in base 10, find n.

JF-1993

5.58 Find the smallest positive integer which cannot be the exterior angle in degrees of a regular polygon.

JS-1993

5.59 Prove that if p and $8p - 1$ are prime numbers then $8p + 1$ is a composite number.

JF-1990

5.60 Find all pairs of integers x, y such that

$$(xy - 1)^2 = (x + 1)^2 + (y + 1)^2.$$

IF-1993

5.61 If $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{c}$ where a, b, c are positive integers with no common factors, prove that $(a + b)$ is a square.

RMO-1992

5.62 Let $p(x) = x^2 + 40$. Show that, for any two integers a, b, either $p(a) + p(b)$ or $p(a) - p(b)$ is composite.

JF-1992

5.63 Let $f(x)$ be a polynomial with integral coefficients. If both $f(1)$ and $f(2)$ are odd prove that for any integer k, $f(k) \neq 0$. JF-1990

5.64 Find all integers n such that $\sqrt{\dfrac{3n-5}{n+1}}$ is also an integer. JF-1992

5.65 Find the number of values of n for which $2^{11}+2^{8}+2^{n}$ is a perfect square. JS-1993

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Combinatorics

6.01 Show that out of 25 given points inside a regular hexagon of side 2, two points can be chosen such that the distance between them is at most 1. JF-1994

6.02 Let $S = \{1, 2, \ldots, 100\}$. If A is a subset of S such that for all $a, b \in A$, $|a - b| \neq 12$, show that $|A| \leq 52$. Can A have 52 elements? IF-1993

6.03 Let $A = \{1, 2, 3, \ldots, 100\}$. Let $B \subseteq A$ such that for all distinct $x, y \in B$, $x + y$ is not divisible by 11. Show that $|B| \leq 47$. INMO-1993

6.04 Let $A = \{(x, y, z) : x, y, z \in \mathbb{N} \text{ and } x + y + z = 12\}$. Find the number of elements in A. IS-1991

6.05 I have six friends and during a certain vacation I met them during several dinners. I found that I dined with all the six exactly on one day; with every five of them on 2 days; with every four on 3 days; with every three on 4 days; with every two on 5 days. Further every friend was present at 7 dinners and every friend was absent at 7 dinners. How many dinners did I have alone? RMO-1993

6.06 There are ten objects with total weight 20, each of the weights being a positive integer. Given that none of the weights exceeds 10, prove that the ten objects can be divided into two groups that balance each other when placed on the two pans of a balance. INMO-1991

6.07 A sergeant appoints arbitrarily chosen 9 or 10 soldiers every night, out of a 33 member guard team, for duty.

Determine the least number of days within which all guards can be on duty the same number of times. How can the guard duty be arranged to achieve this? JF-1993

6.08 Show that the number of ways of distributing 100 identical books into 10 indistinguishable bags so that no two bags contain the same number of books and no bag is empty, is the same as the number of ways of distributing 55 identical books into 10 bags so that no bag is empty.
IF-1994

6.09 Find the number of ways in which one can arrange the numbers $1, 2, \ldots, n^2$ $(n \geq 3)$ as a square matrix of order n such that the numbers in each row and each column are in arithmetic progression. INMO-1992

6.10 Show that there exists a convex hexagon in the plane such that all its interior angles are equal and its sides are 1,2,3,4,5,6 in some order. INMO-1993

6.11* In an examination 70% students of a class failed in Tamil, 75% in English, 80% in Mathematics and 85% in Science. What is the least percentage of the students who failed in all four subjects? JF-1993

6.12 Ten letters are put in 10 addressed envelopes. Find the number of ways in which only nine of the letters are put in the correct envelopes. JS-1993

6.13 Show that the number of 3-element subsets $\{a, b, c\}$ of $\{1, 2, 3, \ldots, 63\}$ with $a+b+c < 95$ is less than the number of those with $a + b + c > 95$. INMO-1995

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* Wording of this problem has been changed.

Solutions

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Algebra

1.01

$$\begin{aligned}
h &= g^{-1}h^2g \\
&= g^{-1}(g^{-1}h^2g)^2g \\
&= g^{-2}h^4g^2 \\
&= g^{-2}(g^{-1}h^2g)^4g^2 \\
&= g^{-3}h^8g^3 \\
&= h^8
\end{aligned}$$

$\Rightarrow$ $h^7 = e$

$\Rightarrow$ order of h is 1 or 7.

1.02 By the A.M.-G.M. inequality, we have

$$3\frac{2x}{3} + 2\frac{5y}{2} + 4\frac{3z}{4} \geq 9\left(\left(\frac{2x}{3}\right)^3\left(\frac{5y}{2}\right)^2\left(\frac{3z}{4}\right)^4\right)^{1/9};$$

$$\begin{aligned}
\text{i.e., } 2x + 5y + 3z &\geq 9\left(\frac{2^3\cdot 5^2\cdot 3^4}{3^3\cdot 2^2\cdot 4^4}\,x^3y^2z^4\right)^{1/9} \\
&= 9\left(\frac{5^2\cdot 3\cdot 7}{2^7}\right)^{1/9} \\
&= 9\left(\frac{525}{2^7}\right)^{1/9}
\end{aligned}$$

1.03

$$\begin{aligned}
ab + bc + cd &\leq ab + bc + cd + ad \\
&= (a+c)(b+d) \\
&\leq \frac{1}{4}\left((a+c)+(b+d)\right)^2 = \frac{1}{4}.
\end{aligned}$$

1.04 Let LHS be $f(x)$. Then $f(a) = f(b) = f(c) = 2$. Therefore $f(x)$, being (apparently) quadratic, is identically equal to 2. Hence the given equation has no roots.

1.05 Since $a,\, b \geq 0$ and $k > 0$

$$
\begin{aligned}
& \sqrt{\sqrt{a+k}-\sqrt{a}} > \sqrt{\sqrt{b+k}-\sqrt{b}} \\
\Longleftrightarrow\quad & \sqrt{a+k}-\sqrt{a} > \sqrt{b+k}-\sqrt{b} \\
\Longleftrightarrow\quad & \sqrt{a+k}+\sqrt{b} > \sqrt{b+k}+\sqrt{a} \\
\Longleftrightarrow\quad & a+k+b+2\sqrt{a+k}\sqrt{b} > b+k+a+2\sqrt{b+k}\sqrt{a} \\
\Longleftrightarrow\quad & (a+k)b > (b+k)a \\
\Longleftrightarrow\quad & b > a.
\end{aligned}
$$

1.06

$$
\begin{aligned}
x^6+1 &= (x^2+1)(x^4-x^2+1) \\
&\geq 2x(x^4-x^2+1)\ (\because x^2+1 \geq 2x \text{ and} \\
&\quad x^4-x^2+1 = (x^2-1)^2+x^2 > 0) \\
&= 2a.
\end{aligned}
$$

Hence $x^6 \geq 2a-1$.

1.07 By 'componendo and dividendo' rule, the given equation yields $\frac{\sqrt{x+1}}{\sqrt{x-1}} = \frac{3+1}{3-1}$; from this, we get $2\sqrt{x-1} = \sqrt{x+1}$. i.e., $4(x-1) = (x+1)$ and therefore $x = \frac{5}{3}$.

1.08 Since m and n are the roots of the first equation, we have $(x+p)(x+q)-k = (x-m)(x-n)$; i.e., $(x+p)(x+q) = (x-m)(x-n)+k$. Therefore $-p$ and $-q$ are the roots of the second equation.

1.09 Let us show that the real roots of

$$x^2+2ax+\frac{1}{16} = -a+\sqrt{a^2+x-\frac{1}{16}} \qquad \cdots (1)$$

are precisely the roots of

$$x^2+2ax+\frac{1}{16} = x. \qquad \cdots (2)$$

Note that the roots of (2) can be easily computed; they are

$$\frac{1}{2} - a \pm \sqrt{a^2 - a + \frac{3}{16}}.$$

Since $0 < a < \frac{1}{4}$,

$$a^2 - a + \frac{3}{16} = \left(\frac{1}{4} - a\right)\left(\frac{3}{4} - a\right) > 0.$$

Therefore the roots are real and distinct; it is not difficult to see that they are positive. Let α be any root of (2). Then

$$(\alpha + a)^2 = a^2 + \alpha - \frac{1}{16}$$

$$\Rightarrow \qquad (\alpha + a) = \sqrt{a^2 + \alpha - \frac{1}{16}}$$

$$\Rightarrow \qquad \alpha = -a + \sqrt{a^2 + \alpha - \frac{1}{16}};$$

$$\text{therefore } \alpha^2 + 2a\alpha + \frac{1}{16} = -a + \sqrt{a^2 + \alpha - \frac{1}{16}};$$

i.e., α is a root of (1). Conversely suppose α is a real root of (1). Let

$$\alpha^2 + 2a\alpha + \frac{1}{16} = \beta. \qquad \cdots (3)$$

Then

$$\beta = (\alpha + a)^2 - a^2 + \frac{1}{16} \geq \frac{1}{16} - a^2 > 0.$$

From (3) and (1) we get

$$\beta = -a + \sqrt{a^2 + \alpha - \frac{1}{16}}$$

$$\Rightarrow \qquad (\beta + a)^2 = a^2 + \alpha - \frac{1}{16}$$

$$\Rightarrow \quad \beta^2 + a^2 + 2a\beta = a^2 + \alpha - \frac{1}{16}$$

$$\Rightarrow \quad \beta^2 + 2a\beta + \frac{1}{16} = \alpha. \qquad \cdots (4)$$

Therefore $\alpha > 0$ and

$$(3) - (4) \Rightarrow (\alpha - \beta)(\alpha + \beta + 2a + 1) = 0.$$

Since α, β, a are positive, $\alpha - \beta = 0$; i.e., $\alpha = \beta$. Therefore by (4), α is a root of (2). Hence the real roots of (1) are the roots of (2), viz.,

$$\frac{1}{2} - a + \sqrt{a^2 - a + \frac{3}{16}} \quad \text{and} \quad \frac{1}{2} - a - \sqrt{a^2 - a + \frac{3}{16}}.$$

1.10 Since

$$\begin{aligned} x + 14 - 8\sqrt{x-2} &= (x-2) - 2 \cdot 4 \cdot \sqrt{x-2} + 16 \\ &= (\sqrt{x-2} - 4)^2 \end{aligned}$$

and

$$\begin{aligned} x + 23 - 10\sqrt{x-2} &= (x-2) - 2 \cdot 5 \cdot \sqrt{x-2} + 25 \\ &= (\sqrt{x-2} - 5)^2 \end{aligned}$$

the given equation is

$$\left|\sqrt{x-2} - 4\right| + \left|\sqrt{x-2} - 5\right| = 3.$$

Let $t = \sqrt{x-2} - 5$. Then we have

$$\begin{aligned} &|t+1| + |t| = 3; \\ &\text{i.e., } (t+1)^2 = (3 - |t|)^2; \\ &\text{i.e., } t + 3\,|t| = 4. \end{aligned}$$

If $t \geq 0$ then $t = 1$ implying $\sqrt{x-2} = 6$, i.e., $x = 38$. Otherwise $-2t = 4$ implying $\sqrt{x-2} = 3$, i.e., $x = 11$. Hence the solution set is $\{11, 38\}$.

1.11 Since $8^2 + 15^2 = 17^2$, the triangle is right angled and therefore its area is $\frac{1}{2} \cdot 8 \cdot 15 = 60$.

1.12 Since $ab = (\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 1$,

$$\begin{aligned}(b+1) + (a+1) &= ab + a + b + 1 \\ &= (a+1)(b+1) \\ \Rightarrow \frac{1}{a+1} + \frac{1}{b+1} &= 1.\end{aligned}$$

1.13 $\frac{b}{4} = \left(\frac{abc}{3 \cdot 4 \cdot 5}\right)^{1/3} = \left(\frac{1620}{3 \cdot 4 \cdot 5}\right)^{1/3} = 3.$

Therefore $b = 4 \cdot 3 = 12$.

1.14 Each of the three fractions is equal to 'the sum of the numerators divided by the sum of the denominators', provided the latter is not zero. Hence each is equal to

$$\frac{a + 2b + a}{2b - c + 3a + c + b} = \frac{2(a+b)}{3(a+b)} = \frac{2}{3}.$$

1.15 Suppose $(x_1, x_2, \ldots, x_n)$ is a rearrangement of $(1, 2, \ldots, n)$ such that $|x_1 - 1|, |x_2 - 1|, \ldots, |x_n - 1|$ are all distinct. Let us show that $n \equiv 0$ or $1 \pmod 4$. Now

$$\sum_{i=1}^{n} (x_i - i) = 0. \qquad \cdots (1)$$

Since $|x_1 - 1|, |x_2 - 2|, \ldots, |x_n - n|$ are all distinct, they have to be a rearrangement of $0, 1, \ldots, n-1$; therefore

$$\sum_{i=1}^{n} |x_i - i| = \frac{n(n-1)}{2}. \qquad \cdots (2)$$

Since for any $k \in \mathbb{Z}$, $|k| \equiv k \pmod 2$,

$$\sum_{i=1}^{n} |x_i - i| \equiv \sum_{i=1}^{n} (x_i - i) \pmod 2.$$

Hence it follows from (1) and (2) that $\dfrac{n(n-1)}{2}$ is even. Therefore $4 \mid n(n-1)$; i.e., 4 divides either n or $n-1$.

1.16 With usual notation, we have $b = 2BM = 2\sqrt{117}$, $a^2 + c^2 = b^2$ and $a + c = 30$. $\therefore$ $2ac = (a+c)^2 - (a^2 + c^2) = 900 - 468 = 432$. Hence the area of the triangle is $\frac{1}{2}ac = 108$.

1.17 $4x^2 = (x-1)^2 = x^2 - 2x + 1 \Rightarrow 3x^2 + 2x - 1 = 0 \Rightarrow (3x-1)(x+1) = 0$. Hence, x being non-negative, is $\frac{1}{3}$.

1.18 Since

$$|x+1| + 2|x-2| = \begin{cases} -3x + 3 \geq 6 & \text{when} \quad x \leq -1 \\ -x + 5 < 6 & \text{when} \quad -1 < x < 2 \\ 3x - 3 < 6 & \text{when} \quad 2 \leq x < 3 \\ 3x - 3 \geq 6 & \text{when} \quad x \geq 3 \end{cases}$$

it follows that

$$|x+1| + 2|x-2| < 6 \iff -1 < x < 3.$$

1.19 Addition of
$$\begin{aligned} x^3 &= a^3 + b^3 + 3ab(a+b), \\ y^3 &= a^3 + b^3 + 3ab(a\omega + b\omega^2) \\ \text{and} \quad z^3 &= a^3 + b^3 + 3ab(a\omega^2 + b\omega) \end{aligned}$$

yields

$$\begin{aligned} x^3 + y^3 + z^3 &= 3(a^3 + b^3) + 3ab(a+b)(1 + \omega + \omega^2) \\ &= 3(a^3 + b^3) \ (\because\ 1 + \omega + \omega^2 = 0). \end{aligned}$$

Second Solution Since

$$\begin{aligned} x+y+z &= (a+b)(1+\omega+\omega^2) = 0, \\ x^3+y^3+z^3 &= 3xyz \\ &= 3(a+b)\omega(a+b\omega)\omega^2(a+b\omega^2) \\ &= 3(a+b)(a+b\omega)(a+b\omega^2) \\ &= 3(a^3+b^3). \end{aligned}$$

1.20 Since $x+3$, $2x+5$ and $3x+2$ are the sides of a triangle, $x+3+3x+2 > 2x+5$; i.e., $2x > 0$ and hence x is positive. Therefore $2x+5 > x+3$ and consequently either $3x+2$ or $2x+5$ is the hypotenuse; now by Pythagoras Theorem, we have two possibilities given by

$$(3x+2)^2 - (2x+5)^2 = \pm(x+3)^2;$$

i.e., either $2x^2-7x-15=0$ or $3x^2-x-6=0$. Now solving these two equations and taking into account that $x>0$ we find that x is either 5 or $(1+\sqrt{73})/6$.

1.21 Let us show that for all $k \geq 0$, $P_k(x)$ is divisible by $P_0(x) = x^4+x^3+x^2+x+1$. For any $k \geq 0$,

$$\begin{aligned} P_{k+1}(x) - P_k(x) &= x^{5(k+1)+4} - x^{5k+4} \\ &= x^{5k+4}(x^5-1) \\ &= x^{5k+4}(x-1)P_0(x) \end{aligned}$$

and therefore $P_0 \mid P_k \Rightarrow P_0 \mid P_{k+1}$. Hence by induction, our claim follows.

1.22 Let x_1, x_2 be any two reals in $[0,1]$ such that $x_1 < x_2$.

$$\begin{aligned} |f(x_1)-f(x_2)| &\leq |f(x_1)| + |f(x_2)| \\ &= |f(x_1)-f(0)| + |f(x_2)-f(1)| \\ &\qquad (\because\ f(0)=f(1)=0) \\ &\leq x_1 + (1-x_2). \end{aligned}$$

Also $|f(x_1)-f(x_2)| < (x_2-x_1)$.

So, adding we get $2|f(x_1) - f(x_2)| < 1$; therefore

$$|f(x_1) - f(x_2)| < \frac{1}{2} \text{ for all } x_1, x_2 \in [0,1].$$

1.23
$$\begin{aligned} F_1 &= \frac{3}{R_1} + \frac{7}{R_1^2} + \frac{3}{R_1^3} + \frac{7}{R_1^4} + \cdots \\ &= \frac{3}{R_1}\left(1 + \frac{1}{R_1^2} + \cdots\right) + \frac{7}{R_1^2}\left(1 + \frac{1}{R_1^2} + \cdots\right) \\ &= \frac{3/R_1}{1 - 1/R_1^2} + \frac{7/R_1^2}{1 - 1/R_1^2} \\ &= \frac{3R_1 + 7}{R_1^2 - 1}. \end{aligned}$$

Similarly $F_2 = \dfrac{7R_1 + 3}{R_1^2 - 1}$. Again, in base R_2,

$$F_1 = \frac{2R_2 + 5}{R_2^2 - 1} \quad \text{and} \quad F_2 = \frac{5R_2 + 2}{R_2^2 - 1}.$$

Thus we have

$$\frac{3R_1 + 7}{R_1^2 - 1} = \frac{2R_2 + 5}{R_2^2 - 1} \qquad \cdots (1)$$

$$\text{and } \frac{7R_1 + 3}{R_1^2 - 1} = \frac{5R_2 + 2}{R_2^2 - 1} \qquad \cdots (2)$$

Now (1) – (2) and (1) + (2) yield after simplification

$$\frac{R_1 + 1}{R_2 + 1} = \frac{4}{3} \quad \text{and} \quad \frac{R_1 - 1}{R_2 - 1} = \frac{10}{7}.$$

This by componendo and dividendo rule gives

$$\frac{R_1 + R_2 + 2}{R_1 - R_2} = \frac{7}{1} \quad \text{and} \quad \frac{R_1 + R_2 - 2}{R_1 - R_2} = \frac{17}{3}.$$

Therefore $\dfrac{R_1 + R_2 + 2}{R_1 + R_2 - 2} = \dfrac{21}{17}$. This by simplification yields $R_1 + R_2 = 19$.

1.24 Let $a = \sin^2 x$ and $b = \cos^2 x$. Since

$$\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right) \geq \left(\frac{2}{a+b}\right)$$

and $a + b = 1$ we have

$$\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$$

where equality holds only when $a = b = 1/2$. Therefore

$$\begin{aligned}
12 + \frac{1}{2}\sin y &= \left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 \\
&\geq \frac{1}{2}\left\{\left(a+\frac{1}{a}\right) + \left(b+\frac{1}{b}\right)\right\}^2 \\
&= \frac{1}{2}\left(a+b+\frac{1}{a}+\frac{1}{b}\right)^2 \\
&\geq \frac{1}{2}(1+4)^2 = \frac{25}{2}
\end{aligned}$$

with equality only when $a = b = \frac{1}{2}$. So we have $\sin y \geq 1$; hence $\sin y = 1$; this also implies $a = b = \frac{1}{2}$; therefore $\sin^2 x = \frac{1}{2}$; i.e., $\sin x = \pm\frac{1}{\sqrt{2}}$. Thus the solutions are

$$x = \frac{\pi}{4} + \frac{k\pi}{2}, \quad y = \frac{\pi}{2} + 2\ell\pi, \quad k, \ell \in \mathbb{Z}.$$

1.25 Let us prove that

$$(1-a_1)(1-a_2)\cdots(1-a_n) > 1 - (a_1 + \cdots + a_n)$$

where $0 < a_1, a_2, \ldots, a_n < 1$ and $n \geq 2$.

$$(1-a_1)(1-a_2) = 1 - (a_1+a_2) + a_1a_2 > 1 - (a_1+a_2).$$

Assume the result to be true for $n = k$. Then for any $a_1, a_2, a_3, \ldots, a_{k+1} \in (0,1)$,

$$(1-a_1)(1-a_2)\cdots(1-a_k) > 1 - (a_1 + \cdots + a_k)$$

and therefore since $(1-a_{k+1})$ is positive,

$$\begin{aligned}
&(1-a_1)(1-a_2)\cdots(1-a_k)(1-a_{k+1})\\
&> \left[1-(a_1+\cdots+a_k)\right](1-a_{k+1})\\
&= 1-(a_1+\cdots+a_k+a_{k+1})+(a_1+\cdots+a_k)a_{k+1}\\
&> 1-(a_1+\cdots+a_k+a_{k+1}).
\end{aligned}$$

Hence by induction, our claim holds.

1.26 Since $\mathbb{Q}$ is countable, it is enough to prove this result with $\mathbb{Q}$ in place of $\mathbb{N}$. For any $x \in \mathbb{R}$, define $A_x = \{r \in \mathbb{Q} : r < x\}$. Then $\mathcal{C} = \{A_x : x \in \mathbb{R}\}$ satisfies our requirement.

1.27 $u \cdot v \cdot uv = -c$; i.e., $u^2v^2 = -c$. Using this in $u^3v^3 + au^2v^2 + buv + c = 0$ we get $uv(b-c) + c(1-a) = 0$. If $b = c$, then $c = 0$ whence $uv = 0$; otherwise $uv = \dfrac{c(a-1)}{(b-c)}$ and therefore it is rational.

The condition that $a \neq 1$ is necessary; the polynomial $x^3 + x^2 - 2x - 2$ satisfies all other conditions of the hypothesis but the conclusion does not hold for this.

1.28 Since $b + c > a \Rightarrow a + b + c > 2a$, $1 - a$ is positive; similarly $1 - b$ and $1 - c$ are positive. Therefore

$$(1-a)(1-b)(1-c) > 0;$$

$$\text{i.e.,}\quad 1-(a+b+c)+ab+bc+ca-abc > 0.$$

$$\text{This yields}\quad -2(ab+bc+ca)+2abc < -2.$$

$$\text{We also have}\quad a^2+b^2+c^2+2(ab+bc+ca) = 4.$$

Adding, we get the result.

1.29 If $f(x) = g(x^3) + xh(x^3)$ is divisible by $x^2 + x + 1$ we can write $f(x) = q(x)(x^2 + x + 1)$. Let ω be a non-real cube root of unity. Then $f(\omega) = q(\omega)(\omega^2 + \omega + 1) = 0$ and $f(\omega^2) = q(\omega^2)(\omega + \omega^2 + 1) = 0$. Therefore $g(1) + \omega h(1) = 0$ and $g(1) + \omega^2 h(1) = 0$. So, it follows that $g(1) = h(1) = 0$; therefore both $g(x)$ and $h(x)$ are divisible by $x - 1$.

Alternative Solution We can write $g(x) = (x-1)g'(x)+g(1)$. Then $g(x^3) = (x^3 - 1)g'(x^3) + g(1)$. Similarly we can write $h(x^3) = (x^3 - 1)h'(x^3) + h(1)$. Therefore

$$f(x) = (x^3 - 1)(g'(x^3) + xh'(x^3)) + g(1) + xh(1).$$

Since $x^2 + x + 1$ divides both $f(x)$ and $x^3 - 1$, it also divides $g(1) + xh(1)$. Therefore $g(1) = h(1) = 0$.

1.30 For any non-negative real x, $(1 + x) \geq 2\sqrt{x}$. Therefore multiplying the four inequalities got by replacing x by a, b, c, d yields $(1 + a)(1 + b)(1 + c)(1 + d) \geq 16\sqrt{abcd} = 16$.

1.31 Suppose that the number of sheep in the flock is $10a + b$ where $0 \leq b \leq 9$. Then the total cash received is $100a^2 + 20ab + b^2$. Since the money remained, was less than rupees 10, when it was the younger brother's turn, we must have $b^2 = 10t + r$ where t is odd and r is the money he took finally; further $b^2 \in \{0, 1, 4, 16, 25, 36, 49, 64, 81\}$; therefore it follows that b^2 is either 16 or 36. Hence $r = 6$. Let the cost of the knife be rupees x. Since the sharing became fair, after the elder gave the knife to the younger, we have $10 - x = 6 + x$; thus the cost of the knife is rupees 2.

1.32 Let the two pans be of weights P_1, P_2 and lengths l_1, l_2 respectively. Let X be the actual weight of the third object and S be its apparent weight. Then

$$\begin{aligned}(X+P_1)\ell_1 &= (S+P_2)\ell_2\\ \text{and } (S+P_1)\ell_1 &= (X+P_2)\ell_2.\end{aligned}$$

Subtracting, we get $(X-S)\ell_1 = (S-X)\ell_2$;

therefore $X = S$.

So we have $(X+P_1)\ell_1 = (X+P_2)\ell_2$. $\cdots$ (1)

Now let Y be the actual weight of the first object. Then

$$(Y+P_1)\ell_1 = (Q_1+P_2)\ell_2 \qquad \cdots (2)$$

$$(Q_2+P_1)\ell_1 = (Y+P_2)\ell_2 \qquad \cdots (3)$$

Now (1) − (2) and (1) − (3) implies

$$\begin{aligned}(X-Y)\ell_1 &= (X-Q_1)\ell_2\\ \text{and} \quad (X-Q_2)\ell_1 &= (X-Y)\ell_2.\end{aligned}$$

From these two relations, we get

$$\frac{X-Q_1}{X-Q_2} = \frac{\ell_1^2}{\ell_2^2}.$$

Similarly we have

$$\frac{X-R_1}{X-R_2} = \frac{\ell_1^2}{\ell_2^2}.$$

Therefore $\dfrac{X-Q_1}{X-Q_2} = \dfrac{X-R_1}{X-R_2}$; i.e.,

$$X^2 - X(Q_1+R_2) + Q_1R_2 = X^2 - X(Q_2+R_1) + Q_2R_1$$

and hence $X = \dfrac{Q_1R_2 - Q_2R_1}{Q_1 - Q_2 - R_1 + R_2}$.

1.33 We can write $f(x)$ as $(x+k)(x-1)^2$. Therefore we have

$$\begin{aligned}x^3 + ax + b &= (x+k)(x^2 - 2x + 1)\\ &= x^3 + (k-2)x^2 + (1-2k)x + k.\end{aligned}$$

Equating the coefficients of x^2, we get $k = 2$. Therefore $x + 2$ divides $f(x)$ and hence the remainder is 0.

1.34 We have

$$
\begin{aligned}
& \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(a+b+c) = 1 \\
\Rightarrow \quad & \sum \frac{1}{a}(b+c) + 3 = 1 \\
\Rightarrow \quad & \sum bc(b+c) + 2abc = 0 \\
\Rightarrow \quad & (b+c)(c+a)(a+b) = 0.
\end{aligned}
$$

Therefore one of $b + c$, $c + a$ or $a + b$ is zero; so we can assume, by symmetry, that $b = -c$.

Now for any odd integer n, $b^n = -c^n$ and therefore

$$\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} = \frac{1}{a^n} = \frac{1}{a^n + b^n + c^n}.$$

1.35 $(x, y) = (3, 2)$ is an obvious solution. Also, when the given equations hold for some (x, y), we have the chain of equivalence: $x \leq 3 \iff \log_3 x \leq 1 \iff \log_2 y \geq 1 \iff y \geq 2 \iff 2^y \geq 4 \iff 3^x \geq 23 + 4 \iff x \geq 3$. Therefore $x = 3$ and hence $y = 2$. Thus $(x, y) = (3, 2)$ is the only solution.

1.36 Let $A = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ where $p_1, \ldots, p_n$ are primes in increasing order and $\alpha_i \in \mathbb{N}$ for $i = 1, \ldots, n$ be highly divisible. When p_1 is replaced by 2, we get a number which has same number of divisors as A; therefore by the definition of highly divisibility, $p_1 = 2$ and hence A is even.

1.37 $bc = (ca)(ab) = a^2 bc \Rightarrow a^2 = 1$ ($\because$ $b, c \neq 0$). Similarly $b^2 = c^2 = 1$. Therefore the good triplets are $(1, 1, 1)$, $(-1, -1, 1)$, $(1, -1, -1)$ and $(-1, 1, -1)$.

1.38 Let $u = \dfrac{1}{\sqrt{ab}}$. Then

$$a^2 + b^2 = 1 \Rightarrow 2ab \le 1 \Rightarrow u \ge \sqrt{2}.$$

Therefore
$$\left(u + \sqrt{2} - \frac{\sqrt{2}}{u}\right) > 0$$

and hence
$$(u - \sqrt{2})\left(u + \sqrt{2} - \frac{\sqrt{2}}{u}\right) \ge 0$$

$$\Rightarrow \qquad u^2 - 2 - \sqrt{2} + \frac{2}{u} \ge 0$$

$$\Rightarrow \qquad u^2 + \frac{2}{u} \ge 2 + \sqrt{2}.$$

Therefore
$$a + b + \frac{1}{ab} \ge 2\sqrt{ab} + \frac{1}{ab} \ge 2 + \sqrt{2}.$$

Note that equality holds if and only if $a = b = \sqrt{2}$.

1.39 $r + s = -b/a$, $rs = c/a$. So

$$\begin{aligned} (ar + b) + (as + b) &= a(r + s) + 2b = b \\ \text{and} \quad (ar + b)(as + b) &= a^2rs + ab(r + s) + b^2 = ac. \end{aligned}$$

Hence $ar + b$ and $as + b$ are the roots of $x^2 - bx + ac = 0$.

Alternative Solution Put $y = ax + b$. Then

$$\begin{aligned} & ax^2 + bx + c = 0 \\ \Rightarrow \quad & (ax)^2 + b(ax) + ac = 0 \\ \Rightarrow \quad & (y - b)^2 + b(y - b) + ac = 0 \\ \Rightarrow \quad & y^2 + b^2 - 2by + by - b^2 + ac = 0 \\ \Rightarrow \quad & y^2 - by + ac = 0. \end{aligned}$$

Thus if r, s are the roots of $ax^2 + bx + c = 0$, then $ar + b$, $as + b$ are the roots of $x^2 - bx + ac = 0$.

1.40 Let x_1, x_2, x_3, x_4 be the roots of the given equation. Then

$$\sum_{1\le i\le 4} x_i = 8 \quad \text{and} \quad \sum_{1\le i<j\le 4} x_i x_j = 24.$$

Therefore

$$\begin{aligned}
192 &= 3\cdot 64\\
&= 3\Big(\sum_i x_i\Big)^2\\
&= 3\sum_i x_i^2 + 6\sum_{i<j} x_i x_j\\
&= \sum_{i<j}(x_i^2 + x_j^2) + 6\cdot 24\\
&\ge 2\sum_{i<j} x_i x_j + 144\\
&\quad \text{(equality holds only when } x_1 = x_2 = x_3 = x_4.)\\
&= 48 + 144 = 192.
\end{aligned}$$

Therefore $x_1 = x_2 = x_3 = x_4 = 2$ implying

$$b = -(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) = -32.$$

1.41 Let $y = f\left(-\frac{b}{a}\right)$; then

$$f(y) - y = a\left(-\frac{b}{a}\right) + b = 0 \Rightarrow f(y) = y;$$

i.e., y is a fixed point of f. Now let z be any fixed point of f; then

$$az + b = f(f(z)) - f(z) = z - z = 0.$$

Therefore $z = -\frac{b}{a}$. Thus it follows that f has a unique fixed point, viz., $-\frac{b}{a}$.

1.42 $\frac{y}{zx} \div \frac{z}{xy} = \frac{y}{zx} \times \frac{xy}{z} = \frac{(xy)^2}{(zx)^2} = \left(\frac{2}{4}\right)^2 = \frac{1}{4}.$

1.43 The given equation can be considered as a quadratic in x:

$$3x^2 + (10 - 4y)x + (3y^2 - 10y + 10) = 0.$$

Solving for x, we get

$$x = \frac{1}{6}\left(4y - 10 \pm \sqrt{(10 - 4y)^2 - 12(3y^2 - 10y + 10)}\right)$$

Since x, y are real

$$(10 - 4y)^2 - 12(3y^2 - 10y + 10) \geq 0$$

which on simplification gives $(y - 1)^2 \leq 0$; i.e., $y = 1$ and therefore $x = -1$.

Alternative Solution LHS of the given equation

$$= 3(x^2 + 2x + 1) + 3(y^2 - 2y + 1) + 4(1 + x - y - xy)$$
$$= 3(1 + x)^2 + 3(1 - y)^2 + 4(1 + x)(1 - y)$$
$$= (1 + x)^2 + (1 - y)^2 +$$
$$2\left[(1 + x)^2 + (1 - y)^2 + 2(1 + x)(1 - y)\right]$$
$$= (1 + x)^2 + (1 - y)^2 + 2\left[(1 + x) + (1 - y)\right]^2.$$

Since x, y are real, $1 + x = 0 = 1 - y$; i.e., $(x, y) = (-1, 1)$.

1.44 Let

$$f(x) = (x - a)(x - b)(x - c)$$
$$= x^3 - x^2(a + b + c) + x(ab + bc + ca) - abc.$$

Obviously when $x \leq 0$, $f(x) < 0$. Therefore the roots of f, viz., a, b, c are all positive.

Alternative Proof Suppose $a \leq 0$; then $b + c > -a \geq 0 \Rightarrow -a(b+c) \geq 0 \Rightarrow bc > -a(b+c) \geq 0 \Rightarrow abc \leq 0$. A contradiction. Hence a, b, c are all positive.

1.45 Since the equation has rational coefficients, $3 - \sqrt{2}$ is also a root. Let the other two roots be α, β. Then $\alpha + \beta + 3 + \sqrt{2} + 3 - \sqrt{2} = 5 \Rightarrow \alpha + \beta = -1$. Therefore $c = \alpha\beta(3+\sqrt{2})(3-\sqrt{2}) = 7\alpha\beta \leq 7\frac{(\alpha+\beta)^2}{4} = \frac{7}{4}$. Hence the largest possible value of c is $\frac{7}{4}$, in which case $\alpha = \beta = -\frac{1}{2}$.

1.46 If x, y, z are positive reals, then

$$\frac{x+y}{\sqrt{xy}} \geq 2, \quad \frac{y+z}{\sqrt{yz}} \geq 2 \quad \text{and} \quad \frac{z+x}{\sqrt{zx}} \geq 2.$$

Therefore

$$\frac{(x+y)(y+z)(z+x)}{xyz} \geq 8.$$

Here putting $x = 1 - a$, $y = 1 - b$ and $z = 1 - c$, we get the required inequality.

1.47 Let $x = \frac{1}{a^2}$ and $y = \frac{1}{b^2}$. Then

$$xy = \left(\frac{1}{ab}\right)^2 \geq \left(\frac{2}{a+b}\right)^4 = 16.$$

Therefore

$$\sqrt[3]{11+x^2} + \sqrt[3]{11+y^2} \geq 2\left\{(11+x^2)(11+y^2)\right\}^{1/6}$$
$$\geq 2\left\{11 + xy\right\}^{1/3}$$

(by Cauchy-Schwarz inequality)

$$\geq 2\{11+16\}^{1/3} = 6.$$

1.48 Let $S = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. For any $x = a + b\sqrt{2} \in S$, let $\overline{x}$ denote $a - b\sqrt{2}$. It can be easily checked that for any $x, y \in S$,

$$x + y \in S, \quad \overline{x+y} = \overline{x} + \overline{y}, \quad xy \in S \quad \text{and} \quad \overline{xy} = \overline{x} \cdot \overline{y}.$$

Now suppose the existence of $x, y \in S$ such that $x^{100} + y^{100} = 7 + 5\sqrt{2}$. Then

$$\overline{x}^{100} + \overline{y}^{100} = \overline{x^{100} + y^{100}} = 7 - 5\sqrt{2} < 0.$$

A contradiction because $\overline{x}^{100} + \overline{y}^{100}$ is a sum of 2 squares.

1.49 Adding 1 to all the three given equations we get,

$$\begin{aligned}(x+1)(y+1) &= 24,\\ (y+1)(z+1) &= 32,\\ (z+1)(x+1) &= 48.\end{aligned}$$

Multiplying all the 3 equations we get

$$\begin{aligned}[(x+1)(y+1)(z+1)]^2 &= 24 \times 32 \times 48\\ &= (8 \times 24)^2\\ \Rightarrow \quad (x+1)(y+1)(z+1) &= \pm(8 \times 24).\end{aligned}$$

Taking the positive value, we get

$$x + 1 = 6, \quad y + 1 = 4 \quad \text{and} \quad z + 1 = 8.$$

Taking the negative value, we get

$$x + 1 = -6, \quad y + 1 = -4 \quad \text{and} \quad z + 1 = -8.$$

So (x, y, z) is either $(5,3,7)$ or $(-7, -5, -9)$.

1.50 Since $p(n)p(n+1)$ is of fourth degree in n with leading coefficient 1, m, if it exists, must be of the form $n^2 + tn + s$. Therefore

$$\begin{aligned} & p(0)p(1) = p(s) \\ \Rightarrow \quad & b^2 + ab + b = s^2 + as + b. \\ \Rightarrow \quad & s^2 - b^2 = a(b - s) \\ \Rightarrow \quad & (s - b)(s + b) = a(b - s) \\ \Rightarrow \quad & (s - b)(s + a + b) = 0. \end{aligned}$$

Thus we have $s = b$ or $s = -(a + b)$. Taking $s = b$ we have

$$\begin{aligned} & (n^2 + an + b)\left((n + 1)^2 + a(n + 1) + b\right) \\ & = (n^2 + tn + b)^2 + a(n^2 + tn + b) + b. \end{aligned}$$

Equating the coefficients of n^3, we obtain $t = a + 1$. Taking $m = n^2 + (a + 1)n + b$, it can be routinely verified that $p(n)p(n + 1) = p(m)$ for all $n \in \mathbb{N}$.

Alternative Solution Writing $p(x) = (x + \alpha)(x + \beta)$, we have $\alpha + \beta = a$ and $\alpha\beta = b$. Now

$$\begin{aligned} (n + \alpha)(n + 1 + \beta) &= n^2 + (\alpha + \beta + 1)n + \alpha\beta + \alpha \\ &= n^2 + (a + 1)n + b + \alpha. \end{aligned}$$

Similarly $(n + \beta)(n + 1 + \alpha) = m + \beta$ where $m = n^2 + (a + 1)n + b$. Therefore

$$\begin{aligned} p(n)p(n + 1) &= (n + \alpha)(n + \beta)(n + 1 + \alpha)(n + 1 + \beta) \\ &= (m + \alpha)(m + \beta) = p(m). \end{aligned}$$

1.51 Proof by induction: Result is clearly true for $n = 1$. Assume the relation to be true for all $n \leq m$. To prove the result for $n = m + 1$:

$$\sum_{k=0}^{m+1} \binom{m+1}{k} x^{(k)} y^{(m+1-k)}$$

$$= \sum_{k=0}^{m+1} \left[\binom{m}{k} + \binom{m}{k-1} \right] x^{(k)} y^{(m+1-k)}$$

(Note: $\binom{m}{m+1} = \binom{m}{-1} = 0$.)

$$= \sum_{k=0}^{m} \binom{m}{k} x^{(k)} (y-1)^{(m-k)} y +$$

$$\sum_{k=1}^{m+1} \binom{m}{k-1} x \cdot (x-1)^{(k-1)} y^{(m+1-k)}$$

$$= y \sum_{k=0}^{m} \binom{m}{k} x^{(k)} (y-1)^{(m-k)} +$$

$$x \sum_{\ell=0}^{m} \binom{m}{\ell} (x-1)^{(\ell)} y^{(m-\ell)}$$

$$= y(x+y-1)^{(m)} + x(x-1+y)^{(m)}$$

(by induction hypothesis)

$$= (x+y)(x+y-1)^{(m)}$$

$$= (x+y)^{(m+1)}$$

Alternative Solution By using Leibnitz's theorem for successive differentiation: If u and v are two functions of a variable t, each being differentiable n times then uv also has nth derivative given as follows:

$$\frac{d^n(uv)}{dt^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k u}{dt^k} \cdot \frac{d^{n-k} v}{dt^{n-k}}.$$

Let $u = t^x$, $v = t^y$. Then LHS is

$$(x+y)(x+y-1)\cdots(x+y-n+1)t^{x+y-n}$$
$$= (x+y)^{(n)} t^{x+y-n}$$

RHS is

$$\sum_{k=0}^{n}\binom{n}{k}x(x-1)\cdots(x-k+1)t^{x-k}\times$$

$$y(y-1)\cdots(y-(n-k)+1)t^{y-(n-k)}$$

$$=\left(\sum_{k=0}^{n}\binom{n}{k}x^{(k)}y^{(n-k)}\right)t^{x+y-n}$$

So cancelling t^{x+y-n} we get the required result.

1.52 Since a_1, a_2, a_3, a_4 are integer-roots of $f(x) - 1991$ with integral coefficients,

$$f(x) - 1991 = (x-a_1)(x-a_2)(x-a_3)(x-a_4)g(x)$$

where $g(x)$ is also a polynomial with integral coefficients. Now suppose a is an integer, such that $f(a) = 1993$. then

$$(a-a_1)(a-a_2)(a-a_3)(a-a_4)g(a) = f(a) - 1991 = 2.$$

Therefore at least three of $a-a_1$, $a-a_2$, $a-a_3$, $a-a_4$ are equal to 1 in magnitude; two of these three should have same sign. Hence two of a_1, a_2, a_3, a_4 are equal, a contradiction.

1.53 If (x, y, z) satisfies the given system of equations, then so does $(-x, -y, z)$; since the system is to have only one solution, we must have $x = y = 0$. Thus $a = b = z$ and $z = \pm 2$. If $a = b = 2$ then

$$xyz + z = 2,\ xyz^2 + z = 2 \text{ and } x^2 + y^2 + z^2 = 4.$$

$\therefore$ $xyz(1-z) = 0$. Taking $z = 1$, then we have $xy = 1$ and $x^2 + y^2 = 3$. This has four real solutions. Hence $a = b = 2$ is not a suitable choice.

So let us consider the case that $a = b = -2$. Then the system becomes

$$xyz + z = -2,\ xyz^2 + z = -2 \text{ and } x^2 + y^2 + z^2 = 4.$$

Here again we have $xyz(1 - z) = 0$. Obviously $z \neq 0$; also $z \neq 1$ for otherwise we would have $xy = -3$ and $x^2 + y^2 = 3$ implying $(x + y)^2 = -3$ whence both x and y could not be real. Therefore either x or y is zero. Now it follows that $z = -2 \Rightarrow x = y = 0$.

Thus the given system has a unique solution if and only if $a = b = -2$.

1.54 Put $f(x) = x + g(x)$; then g has the properties:

$$g(-x) = -g(x),$$
$$g(x + 1) = g(x) \text{ and}$$

when $x \neq 0$, $g\left(\dfrac{1}{x}\right) = \dfrac{g(x)}{x^2}$.

We have to show that $g(x) = 0$ for all $x \in \mathbb{R}$.

$$g(0) = -g(0) \Rightarrow g(0) = 0.$$
$$g(-1) = g(-1 + 1) = g(0) = 0.$$

When x is neither 0 nor -1,

$$\begin{aligned} g(x) &= x^2 g\left(\frac{1}{x}\right) \\ &= x^2 g\left(\frac{x+1}{x}\right) \\ &= x^2 \frac{(x+1)^2}{x^2} g\left(\frac{x}{x+1}\right) \\ &= (x+1)^2 g\left(\frac{x}{x+1} - 1\right) \end{aligned}$$

$$\begin{aligned}
&= (x+1)^2 g\left(\frac{-1}{x+1}\right)\\
&= -(x+1)^2 g\left(\frac{1}{x+1}\right)\\
&= -g(x+1)\\
&= -g(x).
\end{aligned}$$

Therefore $g(x) = 0$.

1.55 Let $x = u + 1$, $y = v + 1$ and $z = w + 1$. From the first equation, we get

$$(u+2)(v+2) = 4(v+1) \Rightarrow uv = 2v - 2u.$$

$$\text{Similarly, } vw = 2w - 2v$$

$$\text{and } wu = 2u - 2w.$$

$$\begin{aligned}
\text{Hence}\quad uvw &= 2vw - wu,\\
uvw &= 2wu - 2uv,\\
uvw &= 2uv - 2vw.
\end{aligned}$$

Adding these three relations we get $3uvw = 0$. This shows that $u = v = w = 0$ and therefore $x = y = z = 1$.

1.56 We are given the following:

$$\begin{aligned}
x_1^3x_3 + x_2^3x_1 + x_3^3x_2 &= -8x_1x_2x_3,\\
x_1 + x_2 + x_3 &= 0,\\
x_1x_2 + x_2x_3 + x_3x_1 &= a,\\
x_1x_2x_3 &= -a \text{ and}
\end{aligned}$$

for $i = 1, 2, 3$, $x_i^3 + ax_i + a = 0$.

Now
$$\begin{aligned}
x_1^3 + ax_1 + a &= 0\\
x_2^3 + ax_2 + a &= 0\\
\text{and } x_3^3 + ax_3 + a &= 0
\end{aligned}$$

$$\Rightarrow \quad (x_1^3x_3 + x_2^3x_1 + x_3^3x_2) + a(x_1x_3 + x_2x_1 + x_3x_2) + a(x_3 + x_1 + x_2) = 0.$$

i.e., $8a + a^2 = 0 \Rightarrow a = -8$. So the given equation is $x^3 - 8x - 8 = 0$. One root is -2; the other roots are given by $x^2 - 2x - 4 = 0$; i.e., $x = 1 \pm \sqrt{5}$. So $\{x_1, x_2, x_3\} = \{-2, 1-\sqrt{5}, 1+\sqrt{5}\}$.

1.57

$$\begin{aligned} 6 = x^2 + y^2 + z^2 &\geq x^2 + \frac{1}{2}(y+z)^2 \\ &= x^2 + \frac{1}{2}(4-x)^2 \\ &= \frac{3x^2}{2} - 4x + 8 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & 3x^2 - 8x + 4 \leq 0 \\ \Rightarrow \quad & (3x-2)(x-2) \leq 0 \\ \Rightarrow \quad & \frac{2}{3} \leq x \leq 2. \end{aligned}$$

When x attains any of the extreme values,

$$y = z = \frac{4-x}{2}.$$

Therefore $x = \frac{2}{3} \Rightarrow y = z = \frac{5}{3}$ and $x = 2 \Rightarrow y = z = 1$.

1.58 If $x > 1$, then $y = x^3 + 3x(x^2 - 1) > x^3 > x$. Thus we get $x < y < z < x$, an impossibility. $\therefore\ x \leq 1$. Since $(-x, -y, -z)$ is also a solution, $-x \leq 1$. So let $x = \cos\theta$, with $0 \leq \theta \leq \pi$. Then $y = 4\cos^3\theta - 3\cos\theta = \cos 3\theta$, $z = \cos 9\theta$ and $x = \cos 27\theta$. $\therefore\ \cos\theta - \cos 27\theta = 0 \Rightarrow \sin 13\theta \times \sin 14\theta = 0$. This leads to 27 solutions:

$$\theta = \frac{k\pi}{13},\ k = 0, 1, 2, \ldots, 13 \text{ and } \theta = \frac{k\pi}{14},\ k = 1, 2, \ldots, 13.$$

Thus the solutions are $(x, y, z) = (\cos\theta, \cos 3\theta, \cos 9\theta)$, with θ as above.

1.59 From $\dfrac{1}{u_1u_2} = \dfrac{u_1}{u_2}$, we get $u_1^2 = 1$; i.e., $u_1 = 1$. For any $n \in \mathbb{N}$,

$$\frac{1}{u_{n+1}u_{n+2}} = \frac{u_{n+1}}{u_{n+2}} - \frac{u_n}{u_{n+1}} = \frac{u_{n+1}^2 - u_nu_{n+2}}{u_{n+2}u_{n+1}}.$$

From this we get

$$\forall\, n \in \mathbb{N} \quad u_{n+1}^2 - 1 = u_nu_{n+2} \qquad \cdots (**)$$

In $(**)$ putting $n = 1, 2$ we get $u_2^2 - 1 = u_3$ and $u_3^2 - 1 = 4u_2$ ($\because\ u_4 = 4$). Therefore $(u_2 - 1)^2(u_2 + 1)^2 = 4u_2 + 1$. Now by the chain of the equivalence

$$\begin{aligned} u_2 \geq 2 &\iff (u_2 - 1)^2 \geq 1 \\ &\iff (u_2 + 1)^2 \leq 4u_2 + 1 \\ &\iff u_2^2 \leq 2u_2 \\ &\iff u_2 \leq 2, \end{aligned}$$

it follows that $u_2 = 2$. Writing $(**)$ as

$$\forall\, n \in \mathbb{N} \quad u_{n+2} = \frac{(u_{n+1} - 1)(u_{n+1} + 1)}{u_n},$$

we get by induction that $u_n = n$ for all $n \in \mathbb{N}$.

1.60 If (x, y) is a solution, then so is $(-x, y)$. Since there should be only one solution, $x = 0$. $\therefore\ y + a = 1$ and $y^2 = 1$. So $y = \pm 1 \Rightarrow a = 2$ or 0. When $a = 2$, (1,0) and $(-1, 0)$ are solutions. Therefore we have to consider only the possibility that $a = 0$ whence we have

$$2^{|x|} + |x| = y + x^2 \quad \text{and} \quad x^2 + y^2 = 1.$$

Therefore $x^2, y^2 \le 1 \;\Rightarrow\; x^2 \le |x|$ and $y \le 1$. Now

$$2^{|x|} + |x| \ge 1 + |x| \ge 1 + x^2 \ge y + x^2 \Rightarrow y = 1.$$

Therefore $x = 0$. Thus the given system has a unique solution if and only if $a = 0$.

1.61 Let $\alpha = x + y$ and $\beta = xy$. Then

$$x^2 + y^2 = \alpha^2 - 2\beta \quad \text{and} \quad x^3 + y^3 = \alpha^3 - 3\alpha\beta.$$

Therefore the given equations can be written as

$$\alpha^3 - 3\alpha\beta = 7 \quad \text{and} \quad \alpha^2 + \alpha - \beta = 4.$$

Also note that α is positive, since

$$\frac{1}{2}(x+y)\left[(x-y)^2 + x^2 + y^2\right] = x^3 + y^3 = 7.$$

Now $\beta = \alpha^2 + \alpha - 4$ and

$$\begin{aligned} & \alpha^3 - 3\alpha(\alpha^2 + \alpha - 4) = 7 \\ \Rightarrow\quad & 2\alpha^3 + 3\alpha^2 - 12\alpha + 7 = 0 \\ \Rightarrow\quad & (\alpha - 1)^2(2\alpha + 7) = 0 \\ \Rightarrow\quad & \alpha = 1.\ (\because\ \alpha > 0.) \end{aligned}$$

So $\beta = \alpha^2 + \alpha - 4 = -2$; i.e.,

$$x + y = 1 \quad \text{and} \quad xy = -2.$$

Solving we get $(x, y) = (2, -1)$ or $(-1, 2)$.

1.62 The given relation can be written as

$$x^8 + y^8 + 6 = 8xy. \qquad \cdots (**)$$

Therefore xy is positive. Let us assume first both are positive. Rewriting (**) as

$$\frac{1}{8}(x^8 + y^8 + 1^8 + 1^8 + 1^8 + 1^8 + 1^8 + 1^8)$$
$$= (x^8 \cdot y^8 \cdot 1^8 \cdot 1^8 \cdot 1^8 \cdot 1^8 \cdot 1^8 \cdot 1^8)^{1/8}$$

we get $x^8 = y^8 = 1 \Rightarrow x = y = 1$. Similarly, when both x, y are negative, we must have $x = y = -1$.

1.63 Let us show that for any $k \leq n-1$,

$$\frac{a_k}{a_{k+1}} + \frac{a_{k+1} - a_k}{a_1} < 2. \qquad \cdots (1)$$

From $a_k - a_{k+1} \leq |a_k - a_{k+1}| < 1 < a_{k+1}$, it follows that $a_k < 2a_{k+1}$ and hence

$$\frac{a_k}{a_{k+1}} < 2. \qquad \cdots (2)$$

Also $a_1 > 1$ and $a_{k+1} - a_k \leq |a_{k+1} - a_k| < 1$ implies

$$\frac{a_{k+1} - a_k}{a_1} < 1. \qquad \cdots (3)$$

If $a_{k+1} < a_k$, $\dfrac{a_{k+1} - a_k}{a_1}$ is negative and in this case (2) $\Rightarrow$ (1); otherwise $\dfrac{a_k}{a_{k+1}} \leq 1$ and (3) $\Rightarrow$ (1). Now summing the relations given by (1), we get

$$\sum_{k=1}^{n-1} \frac{a_k}{a_{k+1}} + \frac{a_n - a_1}{a_1} < 2(n-1);$$

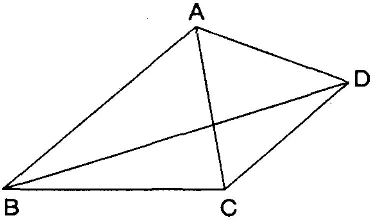
i.e.,
$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} < 2(n-1) + 1$$
$$= 2n - 1.$$

Geometry

2.01 In the usual notation,

$$\frac{1}{2}bc \cdot \sin A = \Delta = \frac{1}{2}bc \cdot \sin 2A = \frac{1}{2}bc \cdot 2 \cdot \sin A \cdot \cos A.$$

Therefore $\cos A = \frac{1}{2} \quad \Rightarrow \quad A = 60^\circ$.

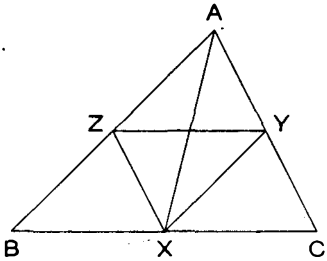


Alternative Proof Let AD be drawn such that $AD = AC$ and $\angle CAD = \angle BAC$. Let $\angle BAC = \theta$. Since $\triangle ABC = \triangle ABD$, $AB \parallel CD$. Now

$$\begin{aligned} \angle ACD &= \theta \quad (\because \; AB \parallel CD). \\ \angle ADC &= \theta \quad (\because \; AC = AD). \\ \angle CAD &= \theta \quad (\text{given}). \end{aligned}$$

Therefore $\triangle ACD$ is equilateral. Thus $\theta = 60^\circ$.

2.02 Let ABC be a triangle and X, Y, Z be the midpoints



of the sides BC, CA, AB respectively. Since $AZXY$ is a parallelogram, its diagonal ZY is bisected by the other diagonal AX; therefore AX is also a median of triangle XYZ. Thus both triangles ABC and XYZ have the same medians and therefore also the same centroid.

Alternative Proof By representation in complex numbers. Let α, β, γ be the affixes of A, B, C respectively. Then the affix of the centroid of $\triangle ABC$ is $\frac{1}{3}(\alpha+\beta+\gamma)$. Now the affixes of X, Y, Z are

$$\frac{\beta+\gamma}{2}, \frac{\gamma+\alpha}{2}, \frac{\alpha+\beta}{2}$$

respectively; therefore the centroid of $\triangle XYZ$ is given by

$$\frac{1}{3}\left(\frac{\beta+\gamma}{2}+\frac{\gamma+\alpha}{2}+\frac{\alpha+\beta}{2}\right)=\frac{1}{3}(\alpha+\beta+\gamma);$$

hence both triangles have the same centroid.

2.03 Proof by representation in complex members. Let z_1, z_2, z_3 be the affixes of A, B, C. Let

$$\frac{BP}{PC}=\frac{CQ}{QA}=\frac{AR}{RB}=\lambda.$$

Then $\frac{z_2+\lambda z_3}{1+\lambda}$, $\frac{z_3+\lambda z_1}{1+\lambda}$ and $\frac{z_1+\lambda z_2}{1+\lambda}$ are respectively the affixes of P, Q and R. We can assume that the circumcenter of $\triangle PQR$ is the origin. Then $\omega(z_2+\lambda z_3) = (z_3+\lambda z_1)$ where $\omega = e^{2i\pi/3}$, a cube root of unity. So we have

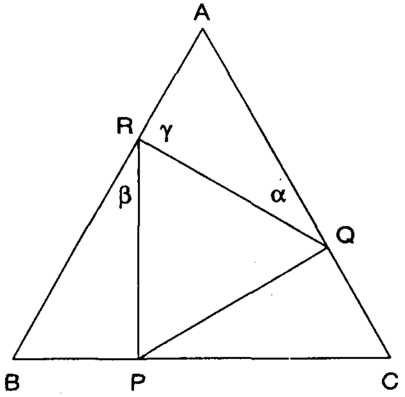
$$\omega z_2 - z_3 = \lambda(z_1 - \omega z_3).$$

Similarly $\omega z_3 - z_1 = \lambda(z_2 - \omega z_1)$

and $\omega z_1 - z_2 = \lambda(z_3 - \omega z_2)$.

$$\begin{aligned} \text{Therefore} \quad \omega z_2 - z_3 &= \lambda(z_1 - \omega z_3) \\ &= \lambda^2(\omega z_1 - z_2) \\ &= \lambda^3(z_3 - \omega z_2) \\ \Rightarrow \quad (1+\lambda^3)(\omega z_2 - z_3) &= 0 \\ \Rightarrow \quad \omega z_2 &= z_3. \\ \text{Similarly} \quad \omega z_3 &= z_1. \end{aligned}$$

Therefore the triangle ABC is equilateral.



Alternative Proof We have

$$\frac{BP}{PC} = \frac{CQ}{QA} = \frac{AR}{RB} = \frac{k}{\ell},$$

where $k + \ell = 1$. By symmetry, it is enough to show that $\angle BAC = \angle PRQ$; this is equivalent to proving $\alpha = \beta$. In $\triangle BPR$,

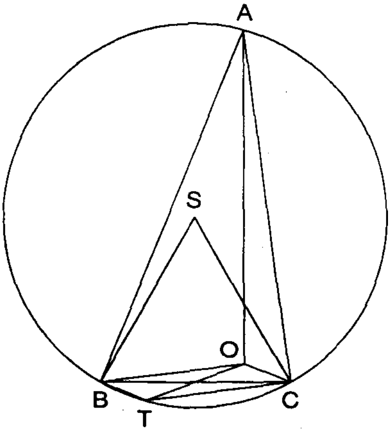
$$\frac{PR}{BP} = \frac{\sin B}{\sin\beta} \Rightarrow PR = BP\frac{\sin B}{\sin\beta} = \frac{ka\sin B}{\sin\beta}.$$

Similarly $QR = \dfrac{\ell b \sin A}{\sin\gamma}$. Since $\dfrac{a}{b} = \dfrac{\sin A}{\sin B}$ and $PR = QR$, we get

$$\frac{\ell}{k} = \frac{\sin\gamma}{\sin\beta} = \frac{\sin(120^\circ - \beta)}{\sin\beta} = \frac{1}{\sin\beta}\left(\frac{\sqrt{3}}{2}\cos\beta + \frac{1}{2}\sin\beta\right)$$
$$= \frac{1}{2}(\sqrt{3}\cot\beta + 1).$$

i.e., $(\sqrt{3}\cot\beta + 1) = \dfrac{2\ell}{k}$. Similarly $(\sqrt{3}\cot\alpha + 1) = \dfrac{2\ell}{k}$. Therefore $\alpha = \beta$.

2.04 The diagonals of the quadrilateral $BTCO$ bisect each other at M. Hence $BTCO$ is a parallelogram. Since $CO \perp AB$ and $CO \parallel TB$, we have $TB \perp AB$. Similarly $TC \perp AC$. Thus the circle having AT as diameter passes through B and C. This is the circumcircle of triangle ABC. Let S be the circumcenter. Then $\angle BSC = 2\angle BAC = 60^\circ$ and $\angle SBC = \angle SCB = 60^\circ$; therefore ΔSBC is equilateral. Therefore BC is equal to the radius of the circle. Therefore $AT = 2BC$.



Alternative Solution Let the circumcenter S of $\triangle ABC$ be the origin. Let z_1, z_2, z_3 be complex numbers representing A, B, C respectively. Then the centroid G is $\dfrac{z_1+z_2+z_3}{3}$. Since G lies on SO such that $SG : SO = 1 : 3$, O is $z_1 + z_2 + z_3$. Since M is the mid-point of BC, its coordinate is $\dfrac{z_2+z_3}{2}$. Since T divides the segment OM in the ratio $2 : -1$, its coordinate is

$$\frac{1}{-1+2}\left(-1(z_1+z_2+z_3)+2\left(\frac{z_2+z_3}{2}\right)\right) = -z_1.$$

$\therefore$ $AT = |2z_1| = 2R$. Also $BC = 2R\sin 30^\circ = R$.

2.05 Suppose (a, b) and (c, d) are two lattice points on a circle with $(\sqrt{2}, \sqrt{3})$ as center. Then

$$\left(a-\sqrt{2}\right)^2 + \left(b-\sqrt{3}\right)^2 = \left(c-\sqrt{2}\right)^2 + \left(d-\sqrt{3}\right)^2.$$

Simplifying we get

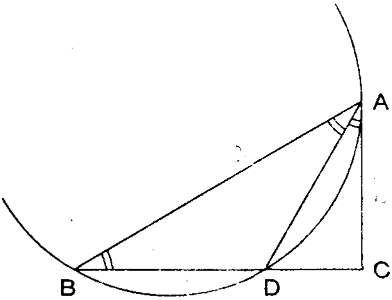
$$a^2 + b^2 - c^2 - d^2 - 2\left(\sqrt{2}(a-c) + \sqrt{3}(b-d)\right) = 0.$$

Therefore it follows that $a = c$ and $b = d$. Hence any circle with $(\sqrt{2}, \sqrt{3})$ as center can pass through at most one lattice point.

Alternative Solution Let P and Q be two lattice points and R be the mid-point of PQ. Let O be the center of the circle. Since the slope of OR is irrational whereas that of PQ is not, OR is not perpendicular to PQ. Hence both P and Q can't be on the circle.

2.06 Let AD be the internal bisector of $\angle A$, meeting BC at D. Then $\dfrac{BD}{DC} = \dfrac{AB}{AC}$; adding 1 to both sides we get

$$\frac{BC}{DC} = \frac{AB+AC}{AC}. \qquad \cdots (1)$$



Since $\angle CAD = \angle ABD$, AC is tangent to the circumcircle of $\triangle ABD$. Therefore

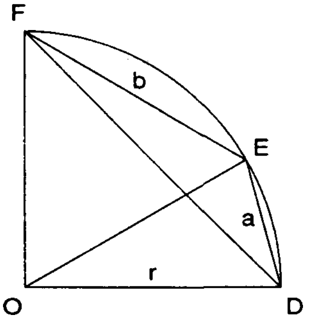
$$BC \cdot DC = AC^2. \qquad \cdots (2)$$

Now (1) × (2) $\Rightarrow$ $BC^2 = AC(AB + AC)$.

Alternative Solution $A = 2B \Rightarrow C = \pi - 3B \Rightarrow \sin C = \sin 3B$. Therefore

$$\begin{aligned} \sin^2 A = \sin^2 2B &= 2\sin B \cos B \sin 2B \\ &= \sin B(\sin B + \sin 3B) \\ &= \sin B(\sin B + \sin C) \\ \Rightarrow \qquad a^2 &= b(b + c). \end{aligned}$$

2.07 Let r be the radius of the circumcircle. Since 4 sides

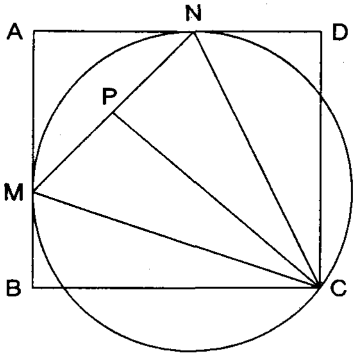


(including DE) are of length a and other sides—EF is one of them—are of length b, $4\angle DOE + 4\angle EOF = 2\pi$; i.e., $\angle DOF = \frac{\pi}{2}$. Now

$$
\begin{aligned}
2\angle DEF &= \angle DEF + \angle OED + \angle OEF \\
&= \angle DEF + \angle ODE + \angle OFE \\
&= 2\pi - \frac{\pi}{2} \\
&= \frac{3}{2}\pi \\
\Rightarrow \qquad \angle DEF &= \frac{3}{4}\pi.
\end{aligned}
$$

$$
\begin{aligned}
\text{Therefore} \quad 2r^2 = FD^2 &= a^2 + b^2 - 2ab\cos\frac{3\pi}{4} \\
&= a^2 + b^2 - 2ab\left(-\frac{1}{\sqrt{2}}\right) \\
&= a^2 + b^2 + \sqrt{2}\,ab \\
\Rightarrow \qquad r &= \sqrt{\frac{a^2 + b^2 + \sqrt{2}\,ab}{2}}.
\end{aligned}
$$

2.08 Let P be the foot of perpendicular from C to MN.



Since BM is a tangent to the circle,

$$\angle BMC = \angle MNC$$

$$\Rightarrow \qquad \sin\angle BMC = \sin\angle MNC$$

$$\Rightarrow \qquad \frac{BC}{CM} = \frac{CP}{CN}$$

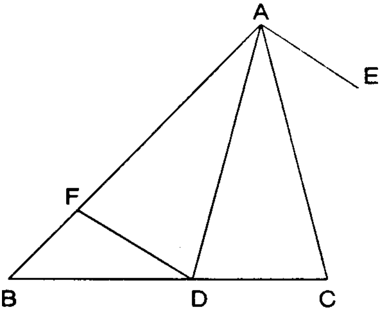
$$\Rightarrow \qquad \frac{BC}{CP} = \frac{CM}{CN}.$$

Similarly $\qquad \dfrac{CD}{CP} = \dfrac{CN}{CM}.$

Multiplying these two relations, we get $BC \cdot CD = CP^2 = 25$; i.e., the area of the rectangle is 25 sq. units.

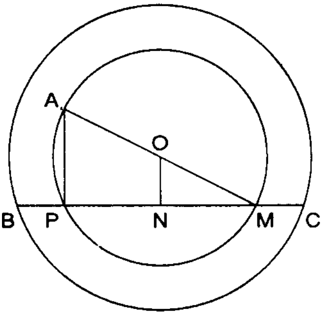
2.09 Let AE be the tangent to the circumcircle at A and DF be drawn parallel to AE meeting AB at F. Then

$$\begin{aligned}\angle ADC &= \angle ABD + \angle BAD\\ &= \angle EAC + \angle CAD \ (\because AE \text{ is tangent to the circumcircle and } AD \text{ bisects } \angle BAC)\\ &= \angle EAD = \angle ADF \ (\because FD \parallel AE).\end{aligned}$$



Since DC touches the incircle, it follows that DF also touches it.

2.10 Let M be the second point of intersection of BC and the smaller circle. Let N be the foot of the perpendicular



drawn from O, the center of the circles, to BC. Let $ON = \alpha$, $PB = \beta$ and $PN = \gamma$. Then $NM = \gamma$ and $MC = \beta$. Since $\angle APM = 90^\circ$, AM is a diameter; therefore O is the midpoint of AM. Hence $AP = 2ON = 2\alpha$. Now

$$\begin{aligned} PA^2 + PB^2 + PC^2 &= 4\alpha^2 + \beta^2 + (\beta + 2\gamma)^2 \\ &= 4\alpha^2 + 2\left((\beta+\gamma)^2 + \gamma^2\right) \\ &= 2(\alpha^2 + (\beta+\gamma)^2 + \alpha^2 + \gamma^2) \\ &= 2(R^2 + r^2). \end{aligned}$$

2.11 Let the axes a, b be $1, 2 + \sqrt{3}$ respectively. Assume P is in the first quadrant and its parameter is θ. Then m_1, the slope of the tangent at P is

$$-\frac{b}{a}\cot\theta = -(2+\sqrt{3})\cot\theta;$$

m_2, the slope of OP is

$$\frac{b\sin\theta}{a\cos\theta} = (2+\sqrt{3})\tan\theta.$$

Let ϕ be the angle between the tangent at P and OP. Then

$$\tan\phi = \frac{m_1 - m_2}{1 + m_1 m_2}$$

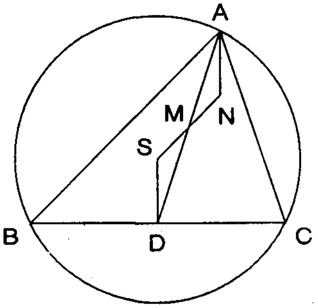
$$= \frac{-(2+\sqrt{3})(\tan\theta+\cot\theta)}{1-(2+\sqrt{3})^2}$$

$$= \frac{\tan\theta+\cot\theta}{2\sqrt{3}} \geq \frac{1}{\sqrt{3}}.$$

$$(\because\ \tan\theta+\cot\theta \geq 2\sqrt{\tan\theta\cdot\cot\theta} = 2.)$$

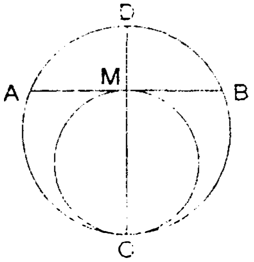
Therefore $\phi \geq \frac{\pi}{6}$. Note that equality holds only when $\tan\theta = 1$; i.e., when $\theta = \frac{\pi}{4}$. If P is not restricted to the first quadrant then $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$.

2.12 Let D be the mid point of BC. Since the diagonals



AD and SN of the quadrilateral $ASDN$ bisect each other, $ASDN$ is a parallelogram. Therefore $AN \parallel SD$. Since $SD \perp BC$, it follows that $AN \perp BC$; i.e., N lies on the altitude through A.

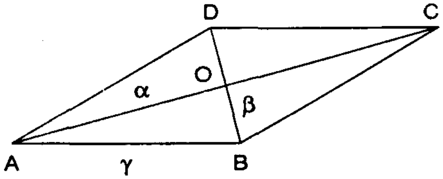
2.13 Let R be the radius of the second circle. The perpendicular to AB at M passes through the centers of the circles, for the respective reasons that AB is a tangent to the inner circle and $MA = MB$. Therefore this line passes



through the point of contact of these circles, say C. Let D be the second point of intersection of this line and the outer circle. Now $CM = 2r$ and $MD = 2(R - r)$ and therefore

$$\begin{aligned} & CM \cdot MD = AM \cdot MB \\ \Rightarrow \quad & 2r \cdot 2(R - r) = a^2 \\ \Rightarrow \quad & R = \frac{a^2 + 4r^2}{4r}. \end{aligned}$$

2.14 Let $\angle OAB$ be θ. Then the area of the rhombus



is $\gamma^2 \sin 2\theta$. It is given that $\gamma^2 \sin 2\theta = \frac{1}{2}\gamma^2$. Therefore $\sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = 30^\circ$ or $150^\circ \Rightarrow \theta = 15^\circ$ or 75°. Hence the required ratio is

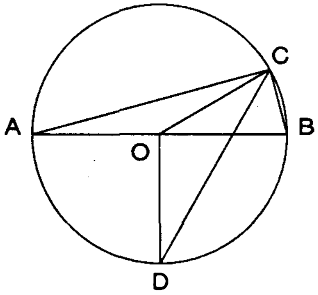
$$\begin{aligned} \tan 75^\circ = \tan(45^\circ + 30^\circ) &= \frac{(1 + 1/\sqrt{3})}{(1 - 1/\sqrt{3})} \\ &= \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \\ &= \frac{(\sqrt{3} + 1)^2}{2} \end{aligned}$$

$$= 2 + \sqrt{3}.$$

Second Solution Areas of the square and the rhombus are $\alpha^2 + \beta^2$ and $2\alpha\beta$ respectively. Therefore

$$\begin{aligned} \alpha^2 + \beta^2 = 4\alpha\beta \quad &\Rightarrow \quad (\alpha+\beta)^2 = 6\alpha\beta \text{ and } (\alpha-\beta)^2 = 2\alpha\beta \\ &\Rightarrow \quad \left(\frac{\alpha+\beta}{\alpha-\beta}\right)^2 = 3 \\ &\Rightarrow \quad \frac{\alpha+\beta}{\alpha-\beta} = \sqrt{3} \\ &\Rightarrow \quad \frac{\alpha}{\beta} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{2} = 2+\sqrt{3}. \end{aligned}$$

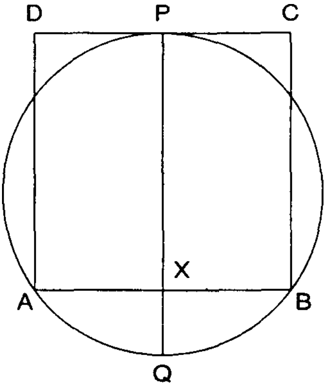
2.15 Let O be the center of the circle. Then $\angle BOC =$



$\dfrac{5\pi}{3} \cdot \dfrac{1}{10} = \dfrac{\pi}{6}$. Since CD bisects $\angle ACB$, D is the mid-point of the arc ADB. Therefore $OD \perp AB$.

Hence $\angle COD = \dfrac{\pi}{6} + \dfrac{\pi}{2} = \dfrac{2\pi}{3}$ and consequently $CD = 2 \cdot 10 \cdot \sin\dfrac{\pi}{3} = 10\sqrt{3}$.

2.16 Let the perpendicular to CD at P the point of contact of CD with the first circle, intersect AB at X and the circle at Q. Then PQ is a diameter; $AX = XB$; $PX = b$ and $XQ = 2r_1 - b$. We know that



$$PX \cdot XQ = AX \cdot BX.$$

i.e., $$b(2r_1 - b) = \left(\frac{1}{2}a\right)^2$$

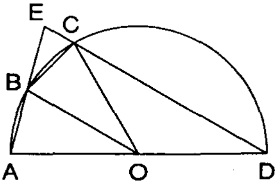
yielding $$8br_1 = a^2 + 4b^2.$$

Similarly $$8ar_2 = b^2 + 4a^2.$$

Therefore $$\begin{aligned} 8ab(r_1 + r_2) &= a^3 + 4ab^2 + b^3 + 4a^2b \\ &= (a+b)^3 + ab(a+b) \\ &= (a^2 + b^2 + 3ab)(a+b) \\ &\geq 5ab(a+b) \end{aligned}$$

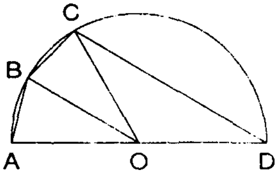
$\Rightarrow$ $$r_1 + r_2 \geq \frac{5}{8}(a+b).$$

2.17 Let AB and DC meet at E. Let $\angle ADE = \theta$; then



$\angle AOC = 2\theta \ \Rightarrow\ \angle AOB = \theta \ \Rightarrow\ OB \parallel DE$. Since O is the midpoint of AB, it follows that $EB = BA = 1$ and $ED = 2BO = 4$. Now $EC \cdot ED = EB \cdot EA \Rightarrow 4EC = 2 \Rightarrow$ $EC = \frac{1}{2}$. Therefore $CD = 4 - \frac{1}{2} = \frac{7}{2}$.

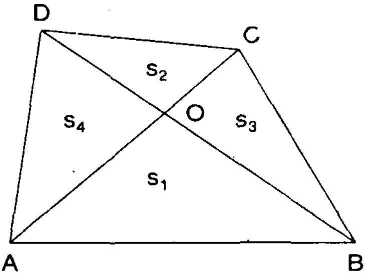
Alternative Proof Let $\angle AOB = \theta$. Then $\angle BOC = \theta$. Therefore



$\angle COD = \pi - 2\theta \Rightarrow CD = 4\sin\left(\frac{\pi - 2\theta}{2}\right) = 4\cos\theta =$

$$4\frac{(OA^2 + OB^2 - AB^2)}{2 \cdot OA \cdot OB} = \frac{7}{2}.$$

2.18 Let s_3 and s_4 be the areas of $\triangle BOC$ and $\triangle AOD$



respectively. Then

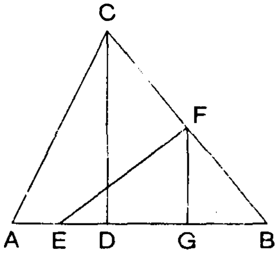
$$\frac{s_1}{s_3} = \frac{AO}{OC} = \frac{s_4}{s_2} \Rightarrow s_1 s_2 = s_3 s_4.$$

$$\begin{aligned} \text{Now}\quad s &= s_1 + s_2 + s_3 + s_4 \\ &\geq s_1 + s_2 + 2\sqrt{s_3 s_4} \\ &= s_1 + s_2 + 2\sqrt{s_1 s_2}. \end{aligned}$$

Therefore $\sqrt{s} \geq \sqrt{s_1} + \sqrt{s_2}$.

$$\begin{aligned}\text{Equality holds} &\iff s_3 = s_4 \\ &\iff s_1 + s_3 = s_1 + s_4 \\ &\iff \triangle ABC = \triangle ABD \\ &\iff AB \parallel CD.\end{aligned}$$

2.19 Let E and F be the mid-points of AD and BC respectively. Draw FG perpendicular to AB. Then G is the mid-point of DB. Therefore



$$EG = ED + DG = \frac{1}{2}(AD + DB) = \frac{1}{2}AB = 4.$$

Also $FG = \frac{1}{2}CD = 3$. Now from the right angled triangle EGF,

$$EF^2 = EG^2 + GF^2 = 4^2 + 3^2 = 5^2$$

and hence $EF = 5$.

Alternate Solution By Analytical Geometry. We can assume that A, D and C are $(2a, 0)$, $(0, 0)$ and $(0, 6)$ respectively. Then B is $(2a + 8, 0)$. Also E, the mid-point of AD is $(a, 0)$ and F, the mid-point of BC is $(a + 4, 3)$. Therefore $EF = \sqrt{4^2 + 3^2} = 5$.

2.20 Let P be (x_1, y_1) in Cartesian coordinates and Q be (r_1, θ_1) in polar coordinates. Then the tangent at P is given by

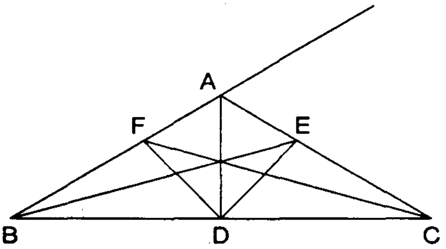
$$xy_1 + yx_1 = 2 \text{ and also by } x\cos\theta_1 + y\sin\theta_1 = r_1.$$

$$\begin{aligned} \text{Therefore} \quad & \frac{r_1}{2} = \frac{\cos\theta_1}{y_1} = \frac{\sin\theta_1}{x_1} \\ \Rightarrow \quad & \frac{r_1^2}{4} = \frac{\cos\theta_1 \sin\theta_1}{y_1 x_1} = \cos\theta_1 \sin\theta_1 \\ \Rightarrow \quad & r_1^2 = 2\sin 2\theta_1. \end{aligned}$$

Hence the locus of Q is $r^2 = 2\sin 2\theta$. Therefore the area enclosed by the curve traced by Q is

$$\frac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/2} 2\sin 2\theta\, d\theta = -\frac{\cos 2\theta}{2}\Big|_0^{\pi/2} = 1.$$

2.21 Let $\angle A = 120^\circ$ and AD, BE, CF be angle bisectors. For $\triangle ABD$, AC is an external bisector and BE is an internal bisector. Hence E is an excenter of $\triangle ABD$. This implies that DE bisects $\angle ADC$. Similarly DF bisects $\angle BDA$. Hence



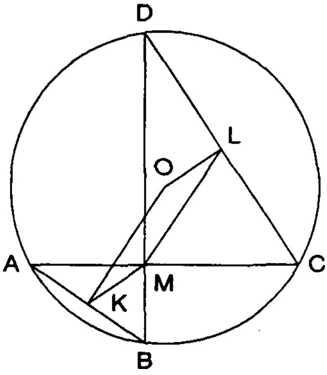
$$\angle FDE = \angle FDA + \angle ADE = \frac{1}{2}\angle BDA + \frac{1}{2}\angle ADC = 90^\circ.$$

2.22 Choose M as the origin. Let A, B, C, D be $(2a, 0)$, $(0, 2b)$, $(2c, 0)$, $(0, 2d)$ respectively. Then

$$AM \cdot MC = BM \cdot MD \Rightarrow ac = bd$$

(in magnitude as well as in sign).

O being the intersection of the perpendicular bisectors of



AC and BD, is $(a+c, b+d)$. Since K, L are (a, b), (c, d) respectively, the midpoint of KL is $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$ which is also the mid-point of OM. Therefore $OKML$ is a parallelogram.

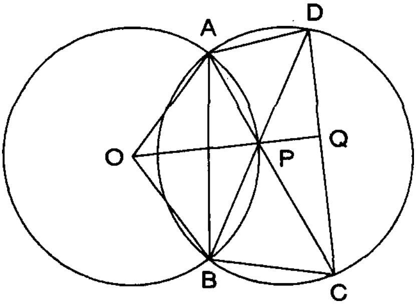
Another Method Let R be the radius of the circle. Then in ΔADB, $AB = 2R\sin\angle ADB$. Also in ΔCAD, $CD = 2R\sin\angle CAD = 2R\cos\angle ADB$ ($\because$ in ΔADM, $\angle ADB + \angle CAD = 90^\circ$). Therefore $AB^2 + CD^2 = 4R^2$. Now

$$AK^2 + OK^2 = AO^2 \Rightarrow \frac{1}{4}AB^2 + OK^2 = R^2$$

and therefore $OK = \frac{1}{2}CD = ML$. ($\because$ in the right angled triangle CMD, L is the mid-point of the hypotenuse.) Similarly $OL = KM$.

Third Method Let $\angle KAM = \theta$. In $\triangle AKM$, $AK = KM$ $\Rightarrow \angle AMK = \theta$. $\therefore \angle AKM = 180^\circ - 2\theta$. So $\angle OKM = 180^\circ - 2\theta - 90^\circ = 90^\circ - 2\theta$. By similar argument, since $\angle MDC = \theta$, we get $\angle OLM = 90^\circ - 2\theta$. Further, $\angle KML = \angle KMA + \angle AMD + \angle DML = \theta + 90^\circ + \theta = 90^\circ + 2\theta$. Thus, three angles $\angle OKM$, $\angle OLM$, $\angle KML$ of $OKML$ are $90^\circ - 2\theta$, $90^\circ - 2\theta$, $90^\circ + 2\theta$ respectively; so the remaining is $90^\circ + 2\theta$ and hence $OKML$ is a parallelogram.

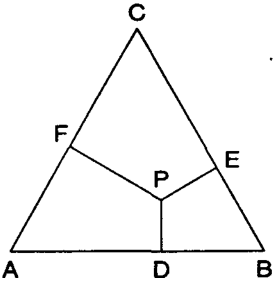
2.23 Let OP produced meet CD at Q. We have to show that OQ is perpendicular to CD.



$$\begin{array}{ll}
\text{Letting} & \angle PDQ = \theta \\
\Rightarrow & \angle BAC = \theta \\
\Rightarrow & \angle POB = 2\theta \\
\Rightarrow & \angle OPB = \dfrac{\pi}{2} - \theta \\
\Rightarrow & \angle DPQ = \dfrac{\pi}{2} - \theta \\
\Rightarrow & \angle DPQ + \angle PDQ = \dfrac{\pi}{2}.
\end{array}$$

Therefore $PQ \perp CD$.

2.24 Since the sum of the areas of the triangles PAB,



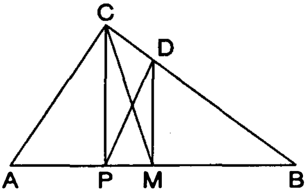
PBC and *PCA* is equal to the area of $\triangle ABC$, we have

$$\frac{1}{2}\cdot 10\cdot(PD+PE+PF)=\frac{\sqrt{3}}{4}\cdot 10^2.$$

$$\therefore\ PD+PE+PF=5\sqrt{3}.$$

So, $PD+PE+PF$ is independent of the position of P.

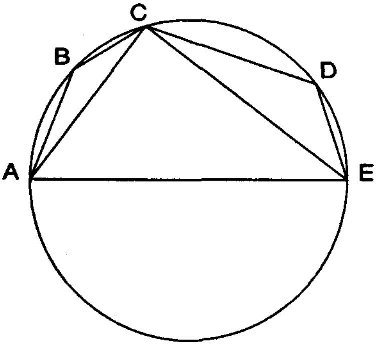
2.25 Since



$$\begin{aligned}\triangle BPD &= \triangle PMD+\triangle MBD\\ &= \triangle CMD+\triangle MBD\ (\because\ PC \parallel MD)\\ &= \triangle CMB\\ &= \frac{1}{2}\triangle ABC,\end{aligned}$$

the ratio under consideration is independent of the position of P.

2.26 Let $\angle CEA=\theta$. Then $\angle ABC=\pi-\theta$. Therefore in $\triangle ABC$,

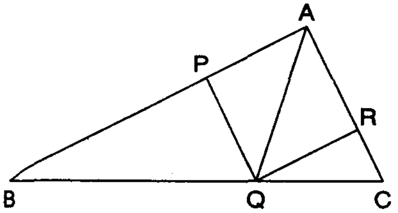


$$
\begin{aligned}
AC^2 &= a^2 + b^2 - 2ab\cos(\pi - \theta) \\
&= a^2 + b^2 + ab \cdot CE \\
&\quad (\because \text{ in } \triangle ACE,\ CE = AE\cos\theta = 2\cos\theta) \\
&> a^2 + b^2 + abc \\
&\quad (\because \angle CDE \text{ is obtuse} \Rightarrow CE > CD).
\end{aligned}
$$

Similarly $CE^2 > c^2 + d^2 + cdb$. Therefore

$$a^2 + b^2 + c^2 + d^2 + abc + bcd < AC^2 + CE^2 = AE^2 = 4.$$

2.27 Let the internal bisector of $\angle A$ meet BC at Q. Let



P, R be points on AB and AC respectively such that $PQ \parallel AC$ and $QR \parallel AB$. Then $APQR$ is a parallelogram. Since AQ bisects $\angle PAR$, $APQR$ is a rhombus. (Alternative way of getting the rhombus: The perpendicular bisector of AQ meets AB and AC at P and R respectively.)

$$\text{Since}\quad \frac{\Delta BPQ}{\Delta ABC} = \frac{BQ^2}{BC^2} \quad (\because\ PQ \parallel AC)$$

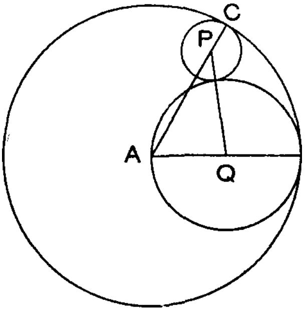
$$\text{and}\quad \frac{\Delta CQR}{\Delta ABC} = \frac{CQ^2}{BC^2} \quad (\because\ QR \parallel AB)$$

$$\text{we have}\ \frac{\Delta BPQ + \Delta CQR}{\Delta ABC} = \frac{BQ^2 + CQ^2}{BC^2}$$

$$\geq \frac{1}{2}\frac{(BQ+QC)^2}{BC^2} = \frac{1}{2}.$$

Therefore the area of the rhombus $\leq \frac{1}{2}\Delta ABC$. (Equality holds only when $BQ = QC$; i.e., when $AB = AC$.)

2.28 We can assume that $AB = 2$. Let Q be the mid point



of AB. Let $PC = x$. Then $AP = 2 - x$; $PQ = 1 + x$. ($\because$ the two circles with centers at P and Q respectively touch each other.) Now in ΔAPQ

$$PQ^2 = AP^2 + AQ^2 - 2AP \cdot AQ \cdot \cos 60^\circ$$

$$\text{i.e.,}\quad (1+x)^2 = (2-x)^2 + 1 - 2(2-x)\frac{1}{2}$$

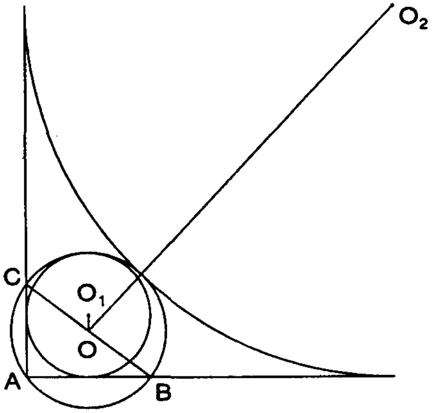
$$\text{i.e.,}\quad 1 + 2x + x^2 = 3 - 3x + x^2.$$

Thus we have $5x = 2$; i.e., $x = \dfrac{AC}{5}$ and hence $AP = \dfrac{4}{5}AC$.

2.29 Let A be the origin; let B and C be $(2b, 0)$ and $(0, 2c)$ respectively. Let O, O_1 and O_2 be the centers of the circles as shown in the figure. O is the midpoint of BC and therefore it is (b, c); if r is the circumradius, then $r^2 = b^2 + c^2$. We know that $OO_1 = r - r_1$; since O_1 is (r_1, r_1), we have

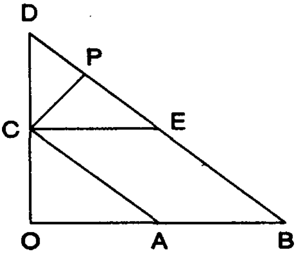
$$(b - r_1)^2 + (c - r_1)^2 = (r - r_1)^2;$$
$$\text{i.e.,}\quad b^2 + c^2 - 2r_1(b + c) + 2r_1^2 = r^2 + r_1^2 - 2rr_1;$$
$$\text{i.e.,}\quad r_1 = 2(b + c - r).$$

Similarly $OO_2 = r + r_2 \Rightarrow r_2 = 2(b+c+r)$. Therefore



$$r_1r_2 = 4\left((b + c)^2 - r^2\right) = 8bc = 4\Delta.$$

2.30 Let A and C be respectively $(a, 0)$ and $(0, b)$. Then



B and D are respectively $(a+4,0)$ and $(0,b+3)$. The equations of the lines AC and BD can be written as

$$x\cos\alpha + y\sin\alpha = p \qquad \cdots (1)$$

$$\text{and} \quad x\cos\alpha + y\sin\alpha = q \qquad \cdots (2)$$

respectively where α is the angle between the x-axis and the perpendicular from O to the parallel lines and p and q are the respectively the distances of the lines AC and BD from the origin. Since $(a,0)$ and $(a+4,0)$ satisfy (1) and (2) respectively, $a\cos\alpha = p$ and $(a+4)\cos\alpha = q$. Therefore

$$4\cos\alpha = q - p \Rightarrow |\cos\alpha| = \frac{d}{4}$$

where d is the distance between AC and BD. Similarly $|\sin\alpha| = \frac{d}{3}$. Therefore

$$\left(\frac{d}{4}\right)^2 + \left(\frac{d}{3}\right)^2 = 1 \Rightarrow d = \frac{12}{5}.$$

Alternative Solution Draw $CE \parallel OA$ meeting BD at E; then $CE = AB = 4$. Let P be the foot of the perpendicular from C to BD; then the distance between AC and BD is CP. In right angled triangle CDE, $CD = 3$ and $CE = 4$ $\Rightarrow$ $DE = 5$. Therefore

$$CP \cdot DE = 2\triangle CDE = CD \cdot CE$$

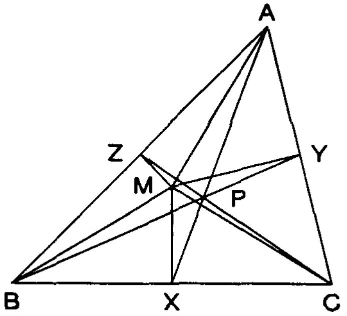
$$\Rightarrow \quad 5CP = 3 \cdot 4; \text{ i.e., } CP = \frac{12}{5}.$$

2.31 Since MX is the internal bisector of $\angle BMC$,

$$\frac{BX}{XC} = \frac{MB}{MC}; \text{ similarly } \frac{CY}{YA} = \frac{MC}{MA} \text{ and } \frac{AZ}{ZB} = \frac{MA}{MB}.$$

Therefore

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$



Hence by Ceva Theorem, AX, BY, CZ are concurrent. Let

$$\triangle BPC = t_1, \quad \triangle APC = t_2 \quad \text{and} \quad \triangle APB = t_3.$$

$$\text{Then} \quad \frac{PA}{PX} = \frac{\triangle APB}{\triangle BPX}$$

$$= \frac{\triangle APC}{\triangle PXC}$$

$$= \frac{\triangle APB + \triangle APC}{\triangle BPX + \triangle PXC}$$

$$= \frac{t_2 + t_3}{t_1}.$$

$$\text{Similarly} \ \frac{PB}{PY} = \frac{t_3 + t_1}{t_2} \ \text{and} \ \frac{PC}{PZ} = \frac{t_1 + t_2}{t_3}.$$

$$\text{Now} \quad 8 = \frac{(t_2 + t_3)(t_3 + t_1)(t_1 + t_2)}{t_1 t_2 t_3}$$

$$\geq \frac{2\sqrt{t_2 t_3} \cdot 2\sqrt{t_3 t_1} \cdot 2\sqrt{t_1 t_2}}{t_1 t_2 t_3}$$

(Equality holds only when $t_1 = t_2 = t_3$.)

$= 8$.

Thus $t_1 = t_2 = t_3$. Now

$$\frac{BX}{XC} = \frac{\triangle ABX}{\triangle ACX}$$

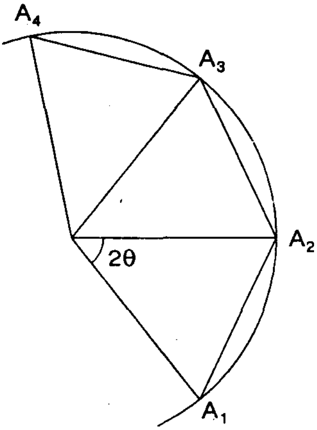
$$= \frac{\triangle PBX}{\triangle PCX}$$

$$= \frac{\triangle ABX - \triangle PBX}{\triangle ACX - \triangle PCX}$$

$$= \frac{t_3}{t_2} = 1.$$

i.e., $BX = XC$. Since MB is the bisector of $\angle BMC$, it follows that $MB = MC$. Similarly $MC = MA$. Therefore M is the circumcenter. (Note that P is the centroid.)

2.32 We can assume that the radius of the circumcircle of the polygon is 1. Let 2θ be the angle subtended by any side of the polygon at the center of the circumcircle. We



know, when a segment of the circle subtends an angle ϕ at the center, its length is $2\sin\frac{\phi}{2}$. Therefore $A_1A_2 = 2\sin\theta$, $A_1A_3 = 2\sin 2\theta$, and $A_1A_4 = 2\sin 3\theta$. Now by hypothesis,

$$\frac{1}{\sin\theta} = \frac{1}{\sin 2\theta} + \frac{1}{\sin 3\theta}$$

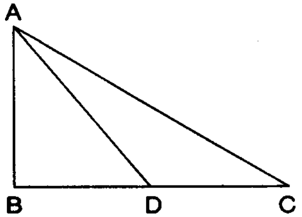
$$\Rightarrow \qquad \frac{1}{\sin\theta} = \frac{\sin 3\theta + \sin 2\theta}{\sin 3\theta \sin 2\theta}$$

$$\Rightarrow \quad \sin 3\theta + \sin 2\theta = 2\sin 3\theta\cos\theta = \sin 4\theta + \sin 2\theta$$

$$\Rightarrow \qquad \sin 3\theta = \sin 4\theta.$$

Since $3\theta, 4\theta \in [0, \pi]$, we get $3\theta + 4\theta = \pi = n\theta \Rightarrow n = 7$.

2.33 Since $AB^2 + BC^2 = AC^2$, $\angle ABC = 90^\circ$.



$$\text{Now} \quad \sin\angle BAC = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \qquad \angle BAC = 60^\circ$$

$$\Rightarrow \qquad \angle BAD = 30^\circ$$

$$\Rightarrow \qquad AD = 3\sec 30^\circ = 2\sqrt{3}.$$

Alternative Solution $\dfrac{BD}{DC} = \dfrac{AB}{AC} = \dfrac{1}{2}$

$$\Rightarrow \quad BD = \frac{BC}{3} = \sqrt{3}$$

$$\text{and} \quad CD = \frac{2BC}{3} = 2\sqrt{3}.$$

Now by Appolonius Theorem, we have

$$2AB^2 + AC^2 = 2BD^2 + DC^2 + 3AD^2;$$

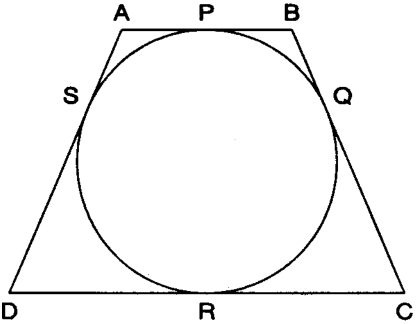
$$\text{i.e., } 2\cdot 3^2 + 6^2 = 2\cdot 3 + 4\cdot 3 + 3AD^2.$$

Simplifying, we get $AD = 2\sqrt{3}$.

2.34 Since the point of concurrence is at the same distance from all four sides, a circle can be drawn having that point as center and touching the sides, as in the figure. Therefore

$$AP = AS,\ BP = BQ,\ CR = CQ \text{ and } DR = DS.$$

Adding these four, we get



$$AB + CD = AD + BC;$$

$$\text{i.e.,}\quad 3 + 7 = AD + 5.$$

$$\text{Hence}\quad AD = 5.$$

Therefore $(AB + BC + CD + DA)/DA = 4$.

2.35 $$\frac{AC}{AR} = 1 + \frac{RC}{AR}$$

$$= 1 + \frac{CT}{AQ} \quad (\because\ CT \parallel AQ)$$

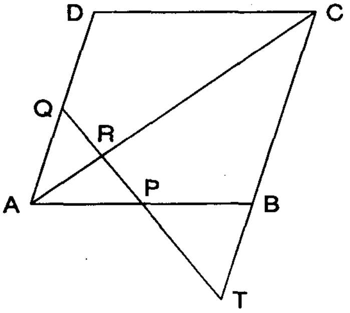
$$= 1 + \frac{BT}{AQ} + \frac{BC}{AQ}$$

$$= 1 + \frac{BP}{AP} + \frac{AD}{AQ}$$

$(\because\ BT \parallel AQ$ and $BC = AD)$

$$= \frac{AB}{AP} + \frac{AD}{AQ}.$$

Alternative Solution Let $\dfrac{AB}{AP} = \alpha$, $\dfrac{AD}{AQ} = \beta$ and $\dfrac{AC}{AR} = \gamma$..

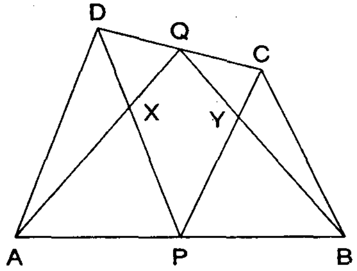


Then $\overline{AB}+\overline{AD}=\overline{AB}+\overline{BC}=\overline{AC}$

$\Rightarrow\ \alpha\overline{AP}+\beta\overline{AQ}=\gamma\overline{AR}$

$\Rightarrow\ \alpha+\beta=\gamma$ ($\because$ P, Q, R are collinear).

2.36 Let d, d_1, d_2 be respectively the distances of the

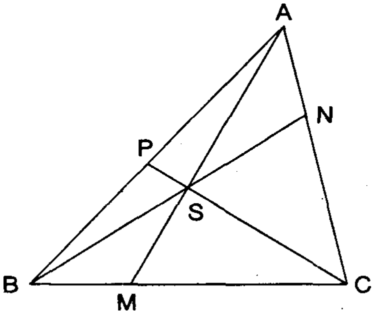


points Q, D, C from AB. Since Q is the midpoint of CD, $2d=d_1+d_2$. Therefore

$$\begin{aligned}\Delta AQB &= \frac{1}{2}AB\cdot d=\frac{1}{2}AB\frac{(d_1+d_2)}{2}\\ &= \frac{1}{2}AB\frac{d_1}{2}+\frac{1}{2}AB\frac{d_2}{2}\\ &= \frac{1}{2}(AP\cdot d_1+BP\cdot d_2)\\ &= \Delta APD+\Delta BPC.\end{aligned}$$

Hence $\triangle AXP + \text{Area of quad. } PXQY + \triangle PYB$
$$= \triangle AXD + \triangle AXP + \triangle PYB + \triangle BYC;$$
i.e., Area of quad. $PXQY = \triangle AXD + \triangle BYC$.

2.37 $$\frac{AS}{AM} = \frac{\triangle ASB}{\triangle ABM} = \frac{\triangle CSA}{\triangle ACM} = \frac{\triangle ASB + \triangle CSA}{\triangle ABM + \triangle ACM}.$$



i.e., $$\frac{R}{AM} = \frac{\triangle ASB + \triangle CSA}{\triangle ABC}.$$

Similarly $$\frac{R}{BN} = \frac{\triangle BSC + \triangle ASB}{\triangle ABC}$$

and $$\frac{R}{CP} = \frac{\triangle CSA + \triangle BSC}{\triangle ABC}.$$

Adding we get $$R\left(\frac{1}{AM} + \frac{1}{BN} + \frac{1}{CP}\right)$$
$$= 2\frac{(\triangle ASB + \triangle BSC + \triangle CSA)}{\triangle ABC} = 2.$$

2.38 By Appolonius Theorem, we have
$$AB^2 + AC^2 = 2(AD^2 + BD^2).$$

This and two similar expressions yield
$$\left.\begin{aligned} 2(b^2 + c^2) &= 4AD^2 + a^2, \\ 2(c^2 + a^2) &= 4BE^2 + b^2 \\ \text{and} \quad 2(a^2 + b^2) &= 4CF^2 + c^2 \end{aligned}\right\} \qquad \cdots (1)$$

Since B, D, G, F are concyclic $AG \cdot AD = AF \cdot AB$; now using $AG = \frac{2}{3}AD$, we get

$$4AD^2 = 3c^2. \qquad \cdots (2)$$

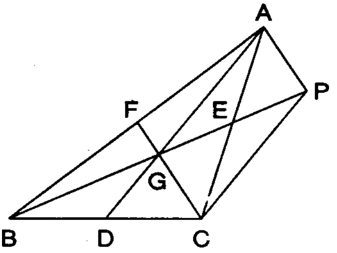
Therefore the first relation of (1) gives

$$2b^2 = c^2 + a^2. \qquad \cdots (3)$$

Since $\angle C$ is obtuse, $c^2 > a^2 + b^2$; therefore by (3),

$$2b^2 > 2a^2 + b^2 \Rightarrow b^2 > 2a^2 \Rightarrow \frac{AC}{BC} > \sqrt{2}.$$

Using (3) in the second and third relations of (1), we get



$4BE^2 = 3b^2$ and $4CF^2 = 3a^2$. Now using these two relations and (2), we get

$$AG = \frac{2}{3}AD = \frac{c}{\sqrt{3}},$$

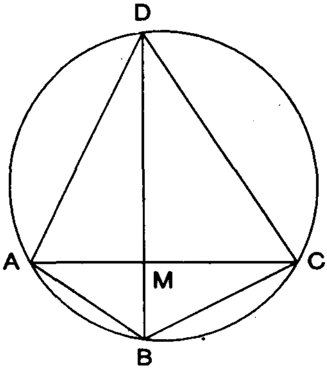
$$GP = 2GE = \frac{2}{3}BE = \frac{b}{\sqrt{3}}$$

$$\text{and} \quad AP = GC = \frac{2}{3}CF = \frac{a}{\sqrt{3}}.$$

Thus $\frac{GA}{AB} = \frac{AP}{BC} = \frac{PG}{CA} = \frac{1}{\sqrt{3}}$. So the triangles GAP and ABC are similar.

2.39 Let $ACB = \theta$; then $AB = 2R\sin\theta$ and $\angle CBD = 90^\circ - \theta \Rightarrow CD = 2R\sin(90^\circ - \theta) = 2R\cos\theta$. Hence

$$\begin{aligned} MA^2 + MB^2 + MC^2 + MD^2 &= AB^2 + CD^2 \\ &= (2R\sin\theta)^2 + (2R\cos\theta)^2 \\ &= 4R^2. \end{aligned}$$



Alternative Solution Choose M as the origin. Let A, B, C, D be $(2a, 0)$, $(0, 2b)$, $(2c, 0)$, $(0, 2d)$ respectively. Then

$$AM \cdot MC = BM \cdot MD \Rightarrow ac = bd$$

(in magnitude as well as in sign).

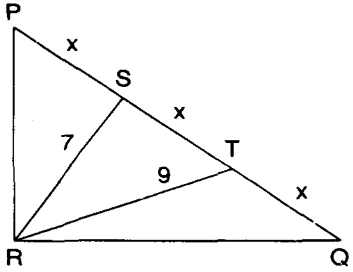
From $(a+c, b+d)$, the point A is at the distance of

$$\sqrt{(a-c)^2 + (b+d)^2} = \sqrt{a^2 + b^2 + c^2 + d^2} \quad (\because\ ac = bd).$$

So are the points B, C, D. Therefore $(a+c, b+d)$ is the center and

$$\begin{aligned} R^2 &= a^2 + b^2 + c^2 + d^2 \\ &= \frac{1}{4}(MA^2 + MB^2 + MC^2 + MD^2). \end{aligned}$$

2.40 Let $ST = x$. Applying Appolonius Theorem to the triangles PRT and ΔQRS, we get



$$PR^2 + 9^2 = 2 \cdot 7^2 + 2x^2 \qquad \cdots (1)$$

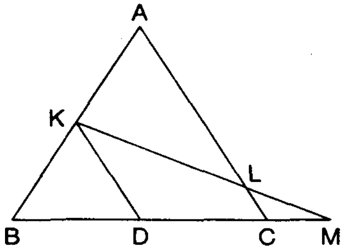
$$QR^2 + 7^2 = 2 \cdot 9^2 + 2x^2 \qquad \cdots (2)$$

Applying Pythagoras Theorem to $\triangle PQR$, we get

$$9x^2 = PR^2 + QR^2 \qquad \cdots (3)$$

Adding (1), (2) and (3) and simplifying we get $x = \sqrt{26}$.

2.41 Draw $KD \parallel LC$, meeting BC at D. Then



$$\angle KDB = \angle LCB = \angle KBD \Rightarrow KB = KD.$$

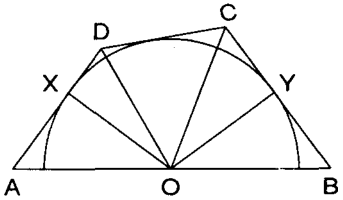
Therefore $\dfrac{KB}{LC} = \dfrac{KD}{LC} = \dfrac{KM}{LM} = 3.$

Alternative Solution The transversal BCM intersects the sides KA, AL, LK of the triangle KAL at B, C, M respectively. Therefore by Menelaus Theorem,

$$\frac{KB}{BA} \cdot \frac{AC}{CL} \cdot \frac{LM}{MK} = -1$$

$$\Rightarrow \qquad \frac{KB}{LC} = 3 \ (\because\ AB = AC \text{ and } KM = 3LM).$$

2.42 Let O be the mid-point of AB. Let X and Y be respectively the points of contact of AD and BC with the circle.



$$\begin{aligned}
\text{Now} \quad & OA = OB, \\
& OX = OY \\
\text{and} \quad & \angle AXO = \angle BYO = 90^\circ \\
\Rightarrow \quad & \triangle OAX \equiv \triangle OBY \\
\Rightarrow \quad & \angle OAD = \angle OBC. \qquad \cdots (1)
\end{aligned}$$

Since DA and DC are tangents,

$$\angle ADC = 2\angle ADO. \qquad \cdots (2)$$

$$\text{Similarly} \quad \angle BCD = 2\angle BCO. \qquad \cdots (3)$$

$$\begin{aligned}
\text{Now} \quad & \angle OAD + \angle ADC + \angle BCD + \angle OBC = 2\pi \\
& = 2(\angle OAD + \angle ADO + \angle AOD) \\
\Rightarrow \quad & \angle BCO = \angle AOD \text{ (by (1), (2) and (3))}.
\end{aligned}$$

This together with (1)

$$\Rightarrow \quad \triangle AOD \cong \triangle BCO$$

$$\Rightarrow \quad \frac{AO}{BC} = \frac{AD}{BO}$$

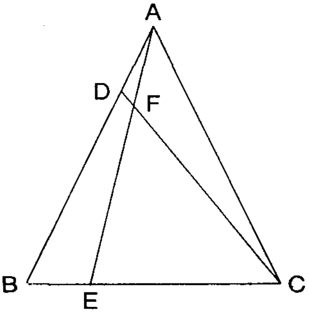
$$\Rightarrow \quad AO \cdot BO = AD \cdot BC$$

$$\Rightarrow \qquad AB^2 = 4 \cdot AD \cdot BC.$$

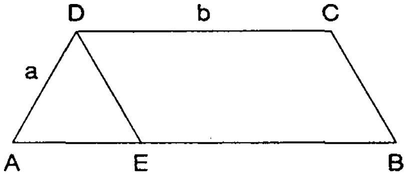
2.43 The transversal EFA intersects the sides BC, CD, DB of the triangle BCD at E, F, A respectively. Therefore by Menelaus Theorem,

$$\frac{BE}{EC} \cdot \frac{CF}{FD} \cdot \frac{DA}{AB} = -1$$

$$\Rightarrow \quad \frac{CF}{FD} = 12 \left(\because \frac{BE}{EC} = \frac{1}{3} \text{ and } \frac{AD}{AB} = \frac{1}{4} \right).$$



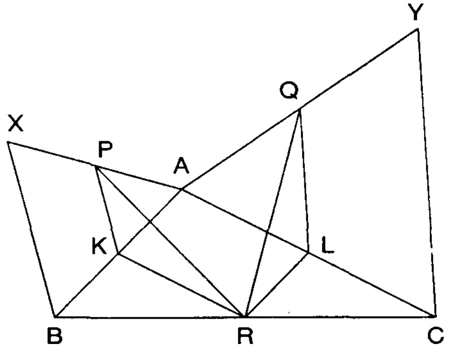
2.44 Let DE be the bisector of $\angle ADC$, meeting AB at E.



Then $\angle ADE = \angle EDC = \angle AED \ \Rightarrow AE = a$. Further $\angle EBC = \angle EDC = \angle AED \ \Rightarrow BC \parallel DE$. Therefore $EBCD$ is a parallelogram so that $EB = b$. Hence $AB = a + b$.

2.45 Let K and L be midpoints of AB and AC respectively. Then $AKRL$ is a parallelogram. Now $PK = KA =$

RL, $KR = AL = LQ$ and $\angle PKR = 60^\circ + \angle AKR = 60^\circ + \angle ALR = \angle QLR$; $\therefore \Delta PKR \equiv \Delta RLQ \Rightarrow PR = QR$ and $\angle KPR = \angle QRL$; also $\angle PRQ = \angle KRL - \angle KRP - \angle QRL = 180^\circ - \angle AKR - \angle KRP - \angle KPR = \angle PKA = 60^\circ$. Therefore ΔPQR is equilateral.



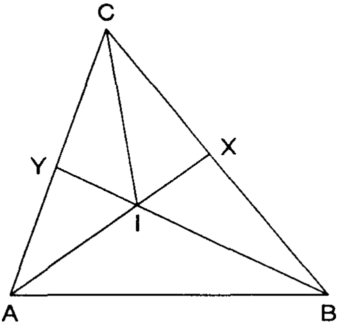
Alternative Solution By representation in complex numbers. Let $\overline{AP} = a$; then $\overline{AB} = 2\omega a$, where $\omega = \cos 60^\circ + i \sin 60^\circ$ ($\because$ rotating $\overline{AX}$ in the anti clockwise direction by 60° we get $\overline{AB}$). Let $\overline{AQ} = b$; then

$$\omega\overline{AC} = \overline{AY} = 2b \Rightarrow \overline{AC} = 2\bar{\omega}b \ (\because \ \bar{\omega} = \omega^{-1}).$$

Therefore $\overline{AR} = \frac{1}{2}(\overline{AB} + \overline{AC}) = \omega a + \bar{\omega}b$.

$$\begin{aligned} \text{Now} \quad RP &= |a - \omega a - \bar{\omega}b| \\ &= |(1-\omega)a - \bar{\omega}b| \\ &= |a - b| \ (\because \ 1 - \omega = \bar{\omega}.) \\ &= PQ. \\ \text{Similarly} \quad QR &= PQ. \end{aligned}$$

2.46 Since $\angle AXC = (A/2 + B)$, we have in ΔCIX,



$$\frac{IC}{IX} = \frac{\sin(A/2+B)}{\sin(C/2)}.$$

Similarly $$\frac{IC}{IY} = \frac{\sin(A+B/2)}{\sin(C/2)}.$$

Now $$IX = IY$$

$\Rightarrow$ $$\sin(A/2+B) = \sin(A+B/2)$$

$\Rightarrow$ $$A/2+B+A+B/2 = \pi$$

$$(\because\ a \neq b \Rightarrow A/2+B \neq A+B/2)$$

$\Rightarrow$ $$3(A+B) = 2\pi$$

$\Rightarrow$ $$3(\pi - C) = 2\pi$$

$\Rightarrow$ $$C = \pi/3.$$

2.47 Let $ABCD$ be a rectangle inscribed in an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a^2 \neq b^2$. Let m_1, m_2 be the slopes of AB and AD respectively. Then

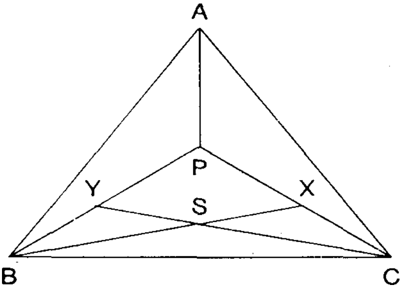
(1) either $m_1 m_2 = -1$ or one of m_1, m_2 is zero.

Let ℓ be the line joining the midpoints of AB and CD; then the slope of ℓ is m_2. Since ℓ is the locus of the midpoints of the chords parallel to AB

(2) either $m_1 m_2 = -\dfrac{b^2}{a^2}$ or one of m_1, m_2 is zero.

Since $b^2 \neq a^2$, by (1) and (2), either m_1 or m_2 is zero. Therefore the sides of $ABCD$ are parallel to the axes.

2.48 Let BY and CX meet at P, a point on AS.



$SA = SB$

$\Rightarrow$ $\angle ABS = 40^\circ$

Therefore $\angle ABP = 20^\circ$

$\Rightarrow$ $\angle APB = 180^\circ - (20^\circ + 40^\circ) = 120^\circ$.

Similarly $\angle APC = 120^\circ$.

Therefore $\angle BPC = 360^\circ - (\angle APB + \angle APC) = 120^\circ$.

Thus $\triangle ABP \equiv \triangle XBP$

$\Rightarrow$ $PA = PX$.

Similarly $PA = PY$.

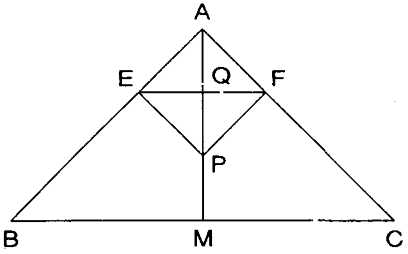
Therefore $\angle APX = \angle XPY = \angle YPA$

and $PA = PX = PY$

$\Rightarrow$ $\triangle AXY$ is equilateral.

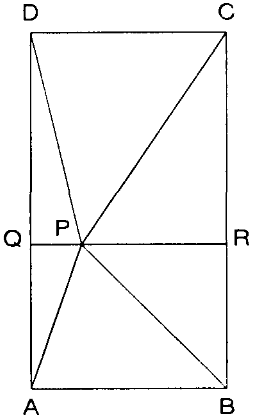
2.49 Let EF and AP intersect at Q. Since $EF \parallel BC$,

$$\frac{EQ}{BM} = \frac{AQ}{AM} = \frac{QF}{MC} \Rightarrow EQ = QF \ (\because\ BM = MC).$$



Since $\angle AEP = \angle AFP = 90^\circ$, EF is a chord of the circle having AP as diameter. Since AP bisects EF, either EF is a diameter in which case $\angle EAF = 90^\circ$ or $AP \perp EF$ in which case $AM \perp BC$ ($\because$ $EF \parallel BC$) implying $AB = AC$.

2.50 Draw a line through P parallel to AB meeting AD and BC at Q and R respectively.

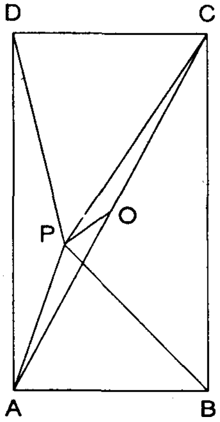


$$\begin{aligned} \text{Then } PA^2 + PC^2 &= AQ^2 + PQ^2 + PR^2 + CR^2 \\ &= BR^2 + PR^2 + PQ^2 + DQ^2 \\ &\quad (\because\ AQ = BR \text{ and } CR = DQ) \\ &= PB^2 + PD^2. \end{aligned}$$

Therefore $3^2 + 5^2 = 4^2 + PD^2$

and hence $PD = 3\sqrt{2}$.

Alternative Solution Let d be the diagonal of $ABCD$ and O be the midpoint of AC. Then in $\triangle PAC$, by Appolonius Theorem



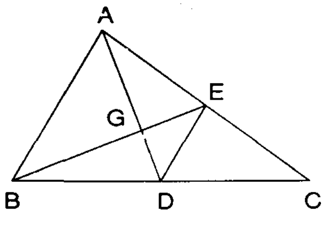
$$\begin{aligned} PA^2 + PC^2 &= 2AO^2 + 2OP^2 \\ &= \frac{1}{2}d^2 + 2OP^2. \end{aligned}$$

Similarly $PB^2 + PD^2 = \frac{1}{2}d^2 + 2OP^2.$

Therefore $PA^2 + PC^2 = PB^2 + PD^2.$

The rest is obvious.

2.51 Applying Pythagoras Theorem to triangles AGB and



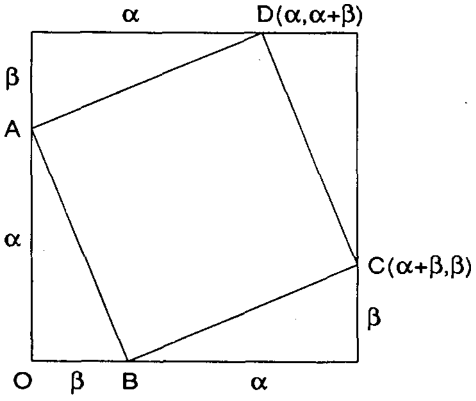
DGE, we get

$$\begin{aligned} AB^2 + DE^2 &= AG^2 + BG^2 + DG^2 + EG^2 \\ &= AG^2 + EG^2 + BG^2 + DG^2 \\ &= AE^2 + BD^2. \end{aligned}$$

$$\text{i.e.,}\quad \frac{5}{4}AB^2 = 9 + \frac{49}{4} \ (\because\ DE = \frac{1}{2}AB).$$

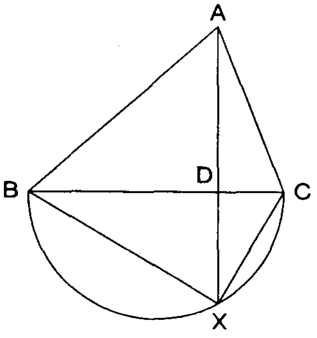
$$\text{i.e.,}\quad AB = \sqrt{17}.$$

2.52 We are given that $(\alpha, \alpha+\beta) = (12, 17)$. Therefore



$\alpha = 12$ and $\beta = 5$; hence C is $(17, 5)$.

2.53 In the right angled triangle BXC, $XD \perp BC$. The-



refore $XD^2 = BD \cdot DC = AD \cot B \cdot AD \cot C$ implying

$$BC^2 \cdot XD^2 = (BC \cdot AD)^2 \cot B \cot C.$$

Therefore $(\Delta BCX)^2 = (\Delta ABC)^2 \cot B \cot C.$

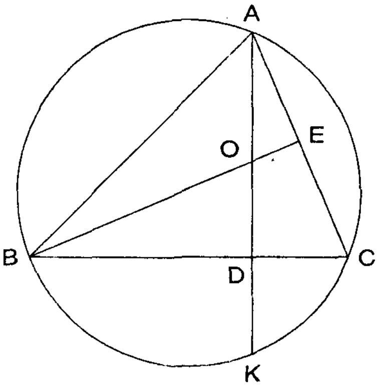
Similarly $(\Delta CAY)^2 = (\Delta ABC)^2 \cot C \cot A$

and $(\Delta ABZ)^2 = (\Delta ABC)^2 \cot A \cot B.$

Since $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$, adding the above three relations we get

$$(\Delta BCX)^2 + (\Delta CAY)^2 + (\Delta ABZ)^2 = (\Delta ABC)^2.$$

2.54 Since



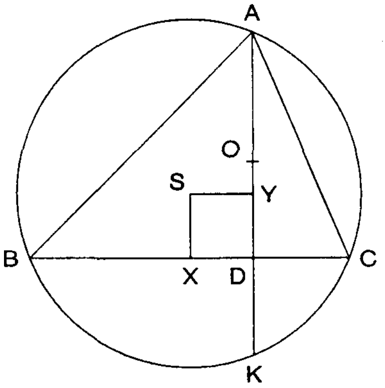
$$\begin{aligned}\angle BCA &= 180^\circ - \angle DOE\\ &= \angle BOD\\ &= \angle BKA\end{aligned}$$

$(\because K$ is the reflection of O in $BC)$,

K lies on the circle passing through A, B, C. Similarly L, M also lie on this circle.

Alternative Solution Let S be the circumcenter of $\triangle ABC$. Let SX and SY be drawn perpendicular to BC and AK respectively. Since

$$AO = 2SX = 2YD,$$



we have
$$\begin{aligned} AY &= AO + OY \\ &= 2YD + OY \\ &= YD + OD \\ &= YD + DK \\ &= YK. \end{aligned}$$

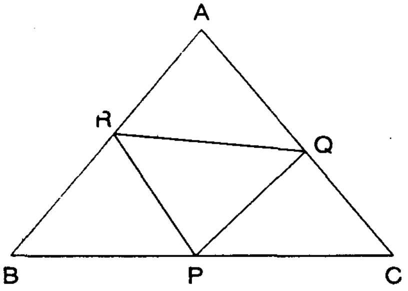
Therefore $SA = SK$

and hence K and similarly L, M lie on the circumcircle of $\triangle ABC$.

❸

Trigonometry

3.01 Let $BP = \alpha BC$, $CQ = \beta CA$ and $AR = \gamma AB$. Then



$$\frac{S_1}{\triangle ABC} = \frac{AR \cdot AQ \cdot \sin A}{AB \cdot AC \cdot \sin A} = (1-\beta)\gamma.$$

We can assume that the area of the triangle ABC is 1; then

$$S_1 = (1-\beta)\gamma.$$
$$\text{Similarly} \quad S_2 = (1-\gamma)\alpha$$
$$\text{and} \quad S_3 = (1-\alpha)\beta.$$

Now $\quad S\left(\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3}\right)$

$$= (1 - S_1 - S_2 - S_3)\left(\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3}\right)$$

$$= \frac{1 - S_2 - S_3}{S_1} + \frac{1 - S_3 - S_1}{S_2} + \frac{1 - S_1 - S_2}{S_3} - 3.$$

Therefore with $\quad \frac{1 - S_2 - S_3}{S_1} = \frac{1 - (1-\gamma)\alpha - (1-\alpha)\beta}{(1-\beta)\gamma}$

$$= \frac{(1-\beta)(1-\alpha) + \alpha\gamma}{(1-\beta)\gamma}$$

$$= \frac{1-\alpha}{\gamma} + \frac{\alpha}{1-\beta}$$

and two more similar evaluations, we get

$$S\left(\frac{1}{S_1}+\frac{1}{S_2}+\frac{1}{S_3}\right)+3$$

$$=\frac{1-\alpha}{\gamma}+\frac{\alpha}{1-\beta}+\frac{1-\beta}{\alpha}+\frac{\beta}{1-\gamma}+\frac{1-\gamma}{\beta}+\frac{\gamma}{1-\alpha}\geq 6.$$

Equality holds only when $\alpha=\beta=\gamma=1/2$.

3.02 $$\sin\frac{\pi}{2n}+\cos\frac{\pi}{2n}=\frac{\sqrt{n}}{2}$$

$$\Rightarrow \quad \sin^2\frac{\pi}{2n}+\cos^2\frac{\pi}{2n}+2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}=\frac{n}{4}$$

$$\Rightarrow \quad 1+\sin\frac{\pi}{n}=\frac{n}{4}$$

$$\Rightarrow \quad \sin\frac{\pi}{n}=\frac{n-4}{4}.$$

Since $0\leq\sin\frac{\pi}{n}\leq 1$, it follows that $4\leq n\leq 8$. A simple verification shows that $n=6$ is the only possibility.

3.03 Since

$$r(a+b+c)=2\Delta=bc\sin A=ca\sin B=ab\sin C,$$

$$\begin{aligned}3r(a+b+c)&=6\Delta\\&=bc\sin A+ca\sin B+ab\sin C\\&<bc+ca+ab\\&\leq\frac{b^2+c^2}{2}+\frac{c^2+a^2}{2}+\frac{a^2+b^2}{2}\\&=a^2+b^2+c^2\end{aligned}$$

$$\Rightarrow \quad r<\frac{a^2+b^2+c^2}{3(a+b+c)}.$$

3.04 Using $a=2R\sin A$ etc., what we have to show becomes

$$\sum_{a,b,c} \frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} \geq 3. \qquad \cdots (**)$$

Put $x = \sqrt{b}+\sqrt{c}-\sqrt{a}$, $y = \sqrt{c}+\sqrt{a}-\sqrt{b}$, $z = \sqrt{a}+\sqrt{b}-\sqrt{c}$.

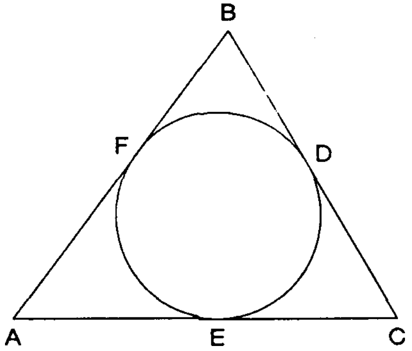
Note that, since $(\sqrt{b}+\sqrt{c})^2 > b+c > a$, x and similarly y, z are positive. Now LHS of $(**)$

$$= \frac{y+z}{2x} + \frac{z+x}{2y} + \frac{x+y}{2z}$$

$$\geq \frac{6}{2}\left\{\frac{y}{x}\cdot\frac{z}{x}\cdot\frac{z}{y}\cdot\frac{x}{y}\cdot\frac{x}{z}\cdot\frac{y}{z}\right\}^{1/6} = 3.$$

Equality holds in $(**)$ only when $a = b = c$.

3.05 Let $BD = x-1$. Then $BF = x-1$, $CD = CE = x$



and $AE = AF = x+1$. Therefore $a = 2x-1$, $b = 2x+1$, $c = 2x$ and $s = 3x$. Now using the formula

$$(s-a)(s-b)(s-c) = sr^2,$$

we get

$$(x+1)(x-1)x = 3x \cdot 16 \Rightarrow x^2 - 1 = 48.$$

Therefore $x^2 = 49 \Rightarrow x = 7$. Thus

$$a = 13,\ b = 15 \text{ and } c = 14.$$

3.06 In the usual notation,

$$\sqrt{s(s-a)(s-b)(s-c)} = \Delta = rs;$$

so by the hypothesis we have

$$(s-a)(s-b)(s-c) = s.$$

Now putting $s - a = x$, $s - b = y$ and $s - c = z$,

$$\begin{aligned}
&\text{we get} && xyz = x + y + z \text{ (Note that } 2x,\ 2y,\ 2z \in \mathbb{N})\\
&\Rightarrow && (2x)(2y)(2z) = 4(2x + 2y + 2z)\\
&\Rightarrow && \text{one of } 2x,\ 2y \text{ or } 2z \text{ is even}\\
&\Rightarrow && 2s \text{ is even}\\
&\Rightarrow && s \in \mathbb{N}\\
&\Rightarrow && x, y, z \in \mathbb{N}.
\end{aligned}$$

Now suppose $x \le y \le z$. Since all three can't be equal, $xyz < 3z$; i.e., $xy \le 2$. From this it follows that $x = 1$, $y = 2$ and $z = 3$. Therefore $a = 5$, $b = 4$ and $c = 3$.

3.07 Using Napier's formula

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2},$$

we get $x = \dfrac{b-c}{b+c}$. Therefore

$$\frac{1+x}{1-x} = \frac{b}{c}. \text{ Similarly, } \frac{1+y}{1-y} = \frac{c}{a} \text{ and } \frac{1+z}{1-z} = \frac{a}{b}.$$

$$\text{Therefore } \frac{(1+x)(1+y)(1+z)}{(1-x)(1-y)(1-z)} = 1.$$

i.e.,
$$1 + (x + y + z) + (xy + yz + zx) + xyz$$
$$= 1 - (x + y + z) + (xy + yz + zx) - xyz.$$

Hence $(x + y + z) + xyz = 0$.

3.08 If $\beta = \cos^{-1}\left(\frac{x}{a}\right)$ and $\gamma = \cos^{-1}\left(\frac{y}{b}\right)$, then $\cos\beta = \frac{x}{a}$ and $\cos\gamma = \frac{y}{b}$. So the problem becomes the following: If $\alpha = \beta + \gamma$, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 + 2\cos\alpha\cos\beta\cos\gamma.$$

Proof
$$\begin{aligned} \text{LHS} &= \cos^2\alpha + \frac{1 + \cos 2\beta}{2} + \frac{1 + \cos 2\gamma}{2} \\ &= 1 + \cos^2\alpha + \cos(\beta + \gamma)\cos(\beta - \gamma) \\ &= 1 + \cos\alpha\cos(\beta + \gamma) + \cos\alpha\cos(\beta - \gamma) \\ &= 1 + \cos\alpha\big(\cos(\beta + \gamma) + \cos(\beta - \gamma)\big) \\ &= 1 + 2\cos\alpha\cos\beta\cos\gamma. \end{aligned}$$

3.09 Let $x = s - a$, $y = s - b$ and $z = s - c$; we know that x, y, z are positive; obviously $a = y + z$, $b = z + x$, $c = x + y$ and $s = x + y + z$. Now the area, circumradius and the inradius are given by

$$\Delta = \sqrt{(x + y + z)(xyz)},\ r = \frac{\Delta}{s} = \frac{\sqrt{xyz}}{\sqrt{x + y + z}}$$

and $R = \dfrac{abc}{4\Delta} = \dfrac{(y + z)(z + x)(x + y)}{4\sqrt{xyz(x + y + z)}}$.

Therefore
$$\begin{aligned} \frac{R}{r} &= \frac{(y + z)(z + x)(x + y)}{4xyz} \\ &= \frac{y(z + x)(x + y) + z(z + x)(x + y)}{4xyz} \\ &= \frac{(z + x)(x + y)}{4zx} + \frac{(z + x)(x + y)}{4xy} \end{aligned}$$

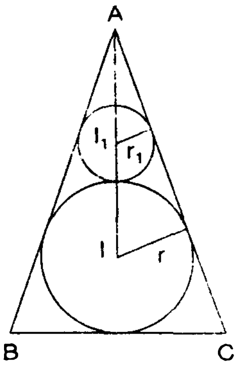
$$
\begin{aligned}
&\geq \frac{(z+x)(x+y)}{(z+x)^2} + \frac{(z+x)(x+y)}{(x+y)^2} \\
&= \frac{(x+y)}{(z+x)} + \frac{(z+x)}{(x+y)} \\
&= \frac{c}{b} + \frac{b}{c}.
\end{aligned}
$$

3.10 Let $\alpha = \cos A$, $\beta = \cos B$ and $\gamma = \cos C$. We know that $\alpha + \beta + \gamma \leq \frac{3}{2}$ with equality only when the triangle is equilateral. Now

$$
\begin{aligned}
27 &= 8(\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma) + 36(\alpha\beta + \beta\gamma + \gamma\alpha) \\
&= 8(\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) + \\
&\quad 36(\alpha\beta + \beta\gamma + \gamma\alpha) \\
&\leq 8 \cdot \frac{3}{2}(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) + \\
&\quad 36(\alpha\beta + \beta\gamma + \gamma\alpha) \\
&= 12(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha) \\
&= 12(\alpha + \beta + \gamma)^2 \leq 27.
\end{aligned}
$$

Therefore $\alpha + \beta + \gamma = \frac{3}{2}$; hence $\triangle ABC$ is equilateral.

3.11 Let r, r_1 be the radii and I, I_1 be the centers of Γ, Γ_1



respectively. Then

$$\sin\frac{A}{2}=\frac{r}{AI}=\frac{r_1}{AI_1}=\frac{r-r_1}{AI-AI_1}=\frac{r-r_1}{r+r_1}.$$

Therefore $\dfrac{(r+r_1)-(r-r_1)}{(r+r_1)+(r-r_1)}=\dfrac{1-\sin(A/2)}{1+\sin(A/2)}$

$$=\frac{1-\cos(\pi/2-A/2)}{1+\cos(\pi/2-A/2)};$$

$$\text{i.e.,}\quad \frac{r_1}{r}=\frac{2\sin^2\left(\frac{\pi-A}{4}\right)}{2\cos^2\left(\frac{\pi-A}{4}\right)}=\tan^2\left(\frac{\pi-A}{4}\right).$$

3.12 Since $\sin A=\dfrac{2\sqrt{s(s-a)(s-b)(s-c)}}{bc}$,

$$\sin B=\frac{2\sqrt{s(s-a)(s-b)(s-c)}}{ca}$$

and $\sin C=\dfrac{2\sqrt{s(s-a)(s-b)(s-c)}}{ab}$,

what we have to prove is equivalent to the following:

$$\frac{s^3(s-a)^3(s-b)^3(s-c)^3}{a^4b^4c^4}\le\frac{3^3}{8^4}.$$

(Equality holds only when $a=b=c$.)

Putting $x=s-a$, $y=s-b$, $z=s-c$, the above inequality becomes the following: If x,y,z are positive reals, then

$$(x+y+z)^3x^3y^3z^3\le\frac{3^3}{8^4}(x+y)^4(y+z)^4(z+x)^4.$$

(Equality holds only when $x=y=z$.)

If α,β,γ are reals, we know that

$$\alpha\beta+\beta\gamma+\gamma\alpha\le\alpha^2+\beta^2+\gamma^2.$$

(Equality holds only when $\alpha=\beta=\gamma$.)

From this it follows that

$$3(\alpha\beta + \beta\gamma + \gamma\alpha) \le (\alpha + \beta + \gamma)^2.$$

Replacing α, β, γ by xy, yz, zx respectively we get

$$3(x + y + z)xyz \le (xy + yz + zx)^2. \qquad \cdots (1)$$

Also $x + y \ge 2\sqrt{xy}$, $y + z \ge 2\sqrt{yz}$ and $z + x \ge 2\sqrt{zx}$
$\Rightarrow \; 8xyz \le (x + y)(y + z)(z + x) \qquad \cdots (2)$

Now

$$\begin{aligned} &3(x + y + z)^3 xyz \\ &\le (xy + yz + zx)^2 (x + y + z)^2 \quad \text{(by (1))} \\ &= \big((x + y)(y + z)(z + x) + xyz\big)^2 \\ &\le (x + y)^2 (y + z)^2 (z + x)^2 \Big(1 + \frac{1}{8}\Big)^2 \quad \text{(by (2))}. \end{aligned}$$

Again by (2),

$$(x + y + z)^3 x^3 y^3 z^3 \le \frac{3^3}{8^4}(x + y)^4 (y + z)^4 (z + x)^2.$$

Equality holds only when $x = y = z$.

Second Solution First let us prove the following: For any three angles α, β and γ such that $\alpha + \beta + \gamma = 2k\pi$ where $k \in \mathbb{Z}$,

$$\sin\alpha + \sin\beta + \sin\gamma \le \frac{3\sqrt{3}}{2}. \qquad \cdots (1)$$

We can choose three unit vectors $\overline{e_1}$, $\overline{e_2}$ and $\overline{e_3}$ such that $\angle(\overline{e_1}, \overline{e_2}) = \alpha$, $\angle(\overline{e_2}, \overline{e_3}) = \beta$ and $\angle(\overline{e_3}, \overline{e_1}) = \gamma$. Then

$$(\overline{e_1} + \overline{e_2} + \overline{e_3}) \cdot (\overline{e_1} + \overline{e_2} + \overline{e_3}) \ge 0$$

$$\Rightarrow \quad 3 + 2(\overline{e_1} \cdot \overline{e_2}) + 2(\overline{e_2} \cdot \overline{e_3}) + 2(\overline{e_3} \cdot \overline{e_1}) \ge 0$$

$$\Rightarrow \qquad \cos\alpha + \cos\beta + \cos\gamma \ge -\frac{3}{2}. \qquad \cdots (2)$$

In (2), α, β, and γ can be replaced by $\alpha + \frac{2\pi}{3}$, $\beta + \frac{2\pi}{3}$ and $\gamma + \frac{2\pi}{3}$ yielding

$$\cos\left(\alpha + \frac{2\pi}{3}\right) + \cos\left(\beta + \frac{2\pi}{3}\right) + \cos\left(\gamma + \frac{2\pi}{3}\right) \geq -\frac{3}{2};$$

$$\text{i.e.,} \qquad -\frac{\sqrt{3}}{2}(\sin\alpha + \sin\beta + \sin\gamma)$$

$$\geq -\frac{3}{2} + \frac{1}{2}(\cos\alpha + \cos\beta + \cos\gamma) \geq -\frac{9}{4} \quad \text{(by (2))}.$$

Thus we get (1). Now putting $\alpha = 2A$, $\beta = 2B$ and $\gamma = 2C$ in (1) and using the fact that

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$$

we get

$$\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}. \qquad \cdots (3)$$

A little work is needed to find when we have equality in (3). When (2) becomes equality, $\overline{e_1} + \overline{e_2} + \overline{e_3} = 0$ from which one can deduce that

$$\overline{e_1} \cdot \overline{e_2} = \overline{e_2} \cdot \overline{e_3} = \overline{e_3} \cdot \overline{e_1};$$

$$\text{i.e.,} \qquad \cos\alpha = \cos\beta = \cos\gamma.$$

Next when equality holds in (1) we should have

$$\cos\left(\alpha + \frac{2\pi}{3}\right) = \cos\left(\beta + \frac{2\pi}{3}\right) = \cos\left(\gamma + \frac{2\pi}{3}\right)$$

$$\text{and} \qquad \cos\alpha = \cos\beta = \cos\gamma,$$

from which it follows that

$$\sin\alpha = \sin\beta = \sin\gamma.$$

Further, equality in (3) yields

$$\sin 2A = \sin 2B = \sin 2C.$$

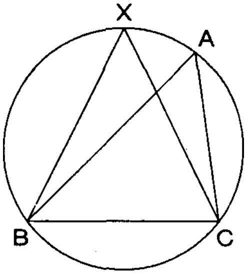
Since $2A + 2B + 2C = 2\pi$ we should have

$$\begin{aligned} 2A + 2B &= \pi \quad \text{or} \quad A = B \\ 2B + 2C &= \pi \quad \text{or} \quad B = C \\ 2C + 2A &= \pi \quad \text{or} \quad C = A. \end{aligned}$$

Again using $A + B + C = \pi$, we get $A = B = C$.

Third Solution For any triangle $\triangle$ let $p(\triangle)$ denote the product of its sides. Let $\mathcal{T}$ be the set of all triangles inscribed in a circle of diameter 1. What we have to prove is equivalent to the following: For any triangle $\triangle \in \mathcal{T}$, $p(\triangle) \leq \frac{3\sqrt{3}}{8}$, equality holding only when $\triangle$ is equilateral.

First let us show that if a triangle $\triangle \in \mathcal{T}$ is not equilateral then there exists a triangle $\triangle'$ in $\mathcal{T}$ such that $p(\triangle) < p(\triangle')$.



Suppose $\triangle ABC \in \mathcal{T}$ such that $AB \neq AC$. Let X be the midpoint of the arc BAC.

Then $\triangle ABC < \triangle XBC$

$\Rightarrow \quad AB \cdot AC \cdot \sin\angle BAC < XB \cdot XC \cdot \sin\angle BXC$

$\Rightarrow \quad p(ABC) < p(XBC)$.

Since $\{p(\triangle) : \triangle \in \mathcal{T}\}$ is a bounded set of real numbers, we can choose a sequence of triangles $(A_nB_nC_n)_{n=1}^{\infty}$ in

$\mathcal{T}$ such that $p(A_nB_nC_n)$ converges to the least upper bound of $\{p(\triangle) : \triangle \in \mathcal{T}\}$ and also A_n, B_n and C_n are convergent. Let $\lim_{n\to\infty} A_n = A$, $\lim_{n\to\infty} B_n = B$ and $\lim_{n\to\infty} C_n = C$. Then A, B, C are on the circle and $p(ABC) = \lim_{n\to\infty} p(A_nB_nC_n)$. Thus $p(ABC)$ being maximum, ABC is equilateral and $p(ABC) = \left(\sin\frac{\pi}{3}\right)^3 = \frac{3\sqrt{3}}{8}$.

Fourth Solution For any $\alpha, \beta, \gamma \in \mathbb{R}$, let

$$\theta(\alpha, \beta, \gamma) = \sin\alpha\sin\beta\sin\gamma.$$

Let us prove the following: If $\alpha, \beta, \alpha', \beta', \gamma \in [0, \pi]$ such that $\alpha \le \beta$, $\alpha', \beta' \in [\alpha, \beta]$ and $\alpha + \beta = \alpha' + \beta'$, then $\theta(\alpha, \beta, \gamma) \le \theta(\alpha', \beta', \gamma)$.

Proof Assume $\alpha' \le \beta'$. Then

$$\begin{aligned}
& 0 \le \alpha \le \alpha' \le \beta' \le \beta \le \pi \\
\Rightarrow\ & 0 \le \beta' - \alpha' \le \beta - \alpha \le \pi \\
\Rightarrow\ & \cos(\beta - \alpha) \le \cos(\beta' - \alpha') \\
\Rightarrow\ & (\because\ \cos x,\ x \in [0, \pi] \text{ is a decreasing function.}) \\
\Rightarrow\ & \cos(\beta - \alpha) - \cos(\alpha + \beta) \le \cos(\beta' - \alpha') - \cos(\alpha' + \beta') \\
\Rightarrow\ & \sin\alpha\sin\beta \le \sin\alpha'\sin\beta' \\
\Rightarrow\ & \theta(\alpha, \beta, \gamma) \le \theta(\alpha', \beta', \gamma).
\end{aligned}$$

Note that equality holds only when either $\{\alpha, \beta\} = \{\alpha', \beta'\}$ or $\gamma \in \{0, \pi\}$. Note that for any $\alpha, \beta, \gamma \in [0, \pi]$,

$$\theta(\alpha, \beta, \gamma) \le \theta\left(\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}, \gamma\right).$$

Now suppose $\alpha, \beta, \gamma \in [0, \pi]$ such that $\alpha + \beta + \gamma = \pi$. We can assume $\alpha \le \frac{\pi}{3} \le \beta$; let $\alpha' = \frac{\pi}{3}$ and $\beta' = \alpha + \beta - \frac{\pi}{3}$; then $\alpha + \beta = \alpha' + \beta'$ and $\alpha', \beta' \in [\alpha, \beta]$. Therefore

$$\begin{aligned}\theta(\alpha,\beta,\gamma) &\le \theta(\alpha',\beta',\gamma)\\ &\le \theta\Big(\alpha',\frac{\beta'+\gamma}{2},\frac{\beta'+\gamma}{2}\Big)\\ &= \theta\Big(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3}\Big)=\frac{3\sqrt{3}}{8}.\end{aligned}$$

Obviously $\theta(\alpha,\beta,\gamma)=\frac{3\sqrt{3}}{8} \iff \alpha=\beta=\gamma=\frac{\pi}{3}$.

Remark The fourth proof leads to the following natural generalization: If $\alpha_1,\alpha_2,\ldots,\alpha_n\in[0,\pi]$ such that $\alpha_1+\alpha_2+\cdots+\alpha_n=\pi$, then

$$\sin\alpha_1\sin\alpha_2\cdots\sin\alpha_n\le\sin^n\Big(\frac{\pi}{n}\Big).$$

Equality holds only when $\alpha_1=\alpha_2=\cdots=\alpha_n$.

3.13 Let S and I be the circumcenter and incenter respectively. We know that

$$SI^2=R^2-2rR \text{ and } \frac{r}{R}=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Therefore the incircle passes through the circumcenter

$$\begin{aligned}&\iff r^2=R^2-2rR\\ &\iff r^2+2rR+R^2=2R^2\\ &\iff r+R=\sqrt{2}R\\ &\iff \frac{r}{R}=(\sqrt{2}-1)\\ &\iff \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{(\sqrt{2}-1)}{4}.\end{aligned}$$

3.14 Let $x=s-a$, $y=s-b$ and $z=s-c$. (Note that x,y,z are positive.) Then

$$\begin{aligned}s&=x+y+z\\ \text{and}\quad \Delta&=\sqrt{(x+y+z)xyz}.\end{aligned}$$

We are given that

$$\begin{aligned} (a+b+c)abc &= 16\Delta^2 \\ \text{i.e., } 2(x+y+z)(x+y)(y+z)(z+x) &= 16(x+y+z)xyz \\ \text{i.e., } (x+y)(y+z)(z+x) &= 8xyz. \end{aligned}$$

Since $x+y \geq 2\sqrt{xy}$, $y+z \geq 2\sqrt{yz}$ and $z+x \geq 2\sqrt{zx}$, we should have $x+y = 2\sqrt{xy}$ and $y+z = 2\sqrt{yz}$. Therefore

$$x = y = z \Rightarrow s-a = s-b = s-c \Rightarrow a = b = c.$$

3.15 From the hypothesis we have

$$\frac{\cos(x+30)\sin x}{\sin(x+30)\cos x} = \frac{\sin(x+10)\sin(x+20)}{\cos(x+10)\cos(x+20)}$$

$$\text{i.e.,} \quad \frac{\sin(2x+30)-\sin 30}{\sin(2x+30)+\sin 30} = \frac{\cos 10-\cos(2x+30)}{\cos 10+\cos(2x+30)}$$

$$\text{i.e.,} \quad \frac{2\sin(2x+30)-1}{2\sin(2x+30)+1} = \frac{\cos 10-\cos(2x+30)}{\cos 10+\cos(2x+30)}$$

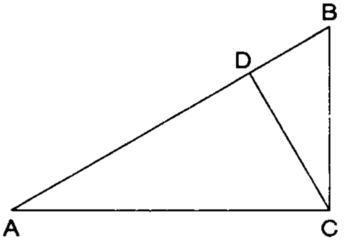
$$\text{i.e.,} \quad 2\sin(2x+30) = \frac{\cos 10}{\cos(2x+30)}$$

$$\text{i.e.,} \quad \sin(4x+60) = \sin 80.$$

$$\therefore \ 4x+60 = \begin{cases} 360k+80 \\ 360k+100 \end{cases} \quad k \in \mathbb{Z}.$$

$$\text{i.e.,} \quad x = 90k+5,\ 90k+10,\ k \in \mathbb{Z}.$$

3.16 Let CD be the shorter trisector of $\angle C$. Then CD is



perpendicular to AB. Therefore

$$\begin{aligned} CD = \sin 60^\circ \cdot BC &= \sin 60^\circ \cdot \cos 60^\circ \cdot AB \\ &= \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \cdot 10 = 5\frac{\sqrt{3}}{2}. \end{aligned}$$

3.17 Put $x = s - a$, $y = s - b$ and $z = s - c$. Then x, y, z are positive and

$$\tan\frac{A}{2} = \sqrt{\frac{yz}{sx}}, \quad \tan\frac{B}{2} = \sqrt{\frac{zx}{sy}} \quad \text{and} \quad \tan\frac{C}{2} = \sqrt{\frac{xy}{sz}}.$$

Now $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}$

$$\begin{aligned}
&\le \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} + \frac{8}{27}\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \\
\iff\quad & \sqrt{\frac{yz}{sx}} + \sqrt{\frac{zx}{sy}} + \sqrt{\frac{xy}{sz}} \le \frac{1}{s}\sqrt{\frac{xyz}{s}} + \frac{8s}{27}\sqrt{\frac{s}{xyz}} \\
\iff\quad & (xy + yz + zx)s \le xyz + \frac{8}{27}s^3 \\
\iff\quad & (xy + yz + zx)(x + y + z) - xyz \le \frac{8}{27}(x + y + z)^3 \\
\iff\quad & xy(x + y) + yz(y + z) + zx(z + x) + 2xyz \\
& \le \frac{8}{27}(x + y + z)^3 \\
\iff\quad & (x + y)(y + z)(z + x) \le \frac{8}{27}(x + y + z)^3.
\end{aligned}$$

Applying G.M.-A.M. inequality to $x + y$, $y + z$ and $z + x$ yields the last relation where equality holds only when $x = y = z$ or equivalently $a = b = c$.

④

Analysis

4.01 Let m be any integer; for any $n \in \mathbb{N}$, since $\dfrac{m}{2^n}$ is the arithmetic mean of 0 and $\dfrac{m}{2^{n-1}}$, using the hypothesis inductively, it follows that $f\left(\dfrac{m}{2^n}\right) = 0$, for all $n \in \mathbb{N}$. Thus f vanishes on $\left\{\dfrac{m}{2^n} : m \in \mathbb{Z},\ n \in \mathbb{N}\right\}$, a dense subset of $\mathbb{R}$. Since f is continuous, $f \equiv 0$.

4.02 The given functional relation can be written as

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2}{x} + \frac{2}{x-1}. \qquad \cdots (1)$$

Let t be any real number other than 0 and 1. Put $x = \dfrac{1}{1-t}$ in (1); then $\dfrac{1}{1-x} = 1 - \dfrac{1}{t}$ and we have

$$f\left(\frac{1}{1-t}\right) + f\left(1 - \frac{1}{t}\right) = -2t + \frac{2}{t}. \qquad \cdots (2)$$

When $x = 1 - \dfrac{1}{t}$, $\dfrac{1}{1-x} = t$ and now (1) yields

$$f\left(1 - \frac{1}{t}\right) + f(t) = \frac{2t}{t-1} - 2t. \qquad \cdots (3)$$

Also replacing x by t in (1) we get

$$f(t) + f\left(\frac{1}{1-t}\right) = \frac{2}{t} + \frac{2}{t-1}. \qquad \cdots (4)$$

Now adding (3) and (4) and subtracting (2) we obtain

$$f(t) = \frac{t+1}{t-1}.$$

4.03 First let us prove the following: Let α be any real number in (0,1). Suppose ϕ is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that for all $x \in \mathbb{R}$

$$\phi(x) - 2\phi(\alpha x) + \phi(\alpha^2 x) = 0. \qquad \cdots (1)$$

Then ϕ is a constant function.

Proof (1) can be written as

$$\phi(x) - \phi(\alpha x) = \phi(\alpha x) - \phi(\alpha^2 x). \qquad \cdots (2)$$

For any $n \in \mathbb{N}$, replacing x in (1) by αx, $\alpha^2 x$, ..., $\alpha^n x$ we get

$$\phi(x) - \phi(\alpha x) = \phi(\alpha^n x) - \phi(\alpha^{n+1} x). \qquad \cdots (3)$$

Since $\alpha^n \to 0$ as $n \to \infty$, by continuity $\phi(\alpha^n x) \to \phi(0)$ as $n \to \infty$. Therefore (3) yields $\phi(x) - \phi(\alpha x) = 0$ i.e., $\phi(x) = \phi(\alpha x)$. So for any $n \in \mathbb{N}$, we have $\phi(x) = \phi(\alpha^n x)$. Therefore

$$\phi(x) = \lim_{n\to\infty} \phi(\alpha^n x) = \phi(0).$$

i.e., ϕ is a constant function. We are given that

$$f(x) - 2f\left(\frac{x}{2}\right) + f\left(\frac{x}{4}\right) = x^2. \qquad \cdots (4)$$

Choosing $\phi(x) = f(x) - \frac{16}{9}x^2$, we see that (4) transforms into (1) for $\alpha = \frac{1}{2}$. Therefore ϕ is a constant function. Hence for some $k \in \mathbb{R}$, $f(x) = k + \frac{16}{9}x^2$.

4.04 Since

$$x_2 = \sqrt{2}^{\sqrt{2}} > \sqrt{2} = x_1$$

and for any integer $n > 1$,

$$x_n > x_{n-1} \Rightarrow \sqrt{2}^{x_n} > \sqrt{2}^{x_{n-1}} \Rightarrow x_{n+1} > x_n,$$

it follows by induction that $(x_n)_{n=1}^{\infty}$ is an increasing sequence. Also by induction it is easy to see that $(x_n)_{n=1}^{\infty}$ is bounded above by 2. Therefore this sequence has a limit, say ℓ. Now the defining relation of the sequence gives

$$\ell = \sqrt{2}^{\ell} = (2^{1/2})^{\ell} = (1+1)^{\ell/2} \geq 1 + \frac{\ell}{2}$$

Therefore $\frac{\ell}{2} \geq 1 \quad \Rightarrow \ell \geq 2$. Hence $\ell = 2$.

4.05 For any $x \in \mathbb{R}$, we have

$$f(x) - f\left(\frac{x}{3}\right) = \frac{2x}{3}$$

$$f\left(\frac{x}{3}\right) - f\left(\frac{x}{3^2}\right) = \frac{2x}{3^2}$$

$$f\left(\frac{x}{3^2}\right) - f\left(\frac{x}{3^3}\right) = \frac{2x}{3^3}$$

$$\vdots$$

$$f\left(\frac{x}{3^{n-1}}\right) - f\left(\frac{x}{3^n}\right) = \frac{2x}{3^n}$$

Adding, we get

$$\begin{aligned} f(x) - f\left(\frac{x}{3^n}\right) &= \frac{2x}{3}\left\{1 + \frac{1}{3} + \cdots + \frac{1}{3^{n-1}}\right\} \\ &= x\left(1 - \frac{1}{3^n}\right). \end{aligned}$$

Since f is continuous,

$$\lim_{n\to\infty} f\left(\frac{x}{3^n}\right) = f(0).$$

Therefore $f(x) - f(0) = x$.

Hence $f(x) = x + f(0) \ \forall\ x \in \mathbb{R}$.

4.06 The binomial identity $\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$ gives on integrating from 0 to n,

$$\sum_{k=0}^{n} \binom{n}{k} \frac{n^{k+1}}{k+1} = \frac{(1+n)^{n+1}-1}{(n+1)}.$$

$$\therefore \qquad \frac{1}{n^n} \sum_{k=0}^{n} \binom{n}{k} \frac{n^{k+1}}{k+1} = \frac{(1+n)^{n+1}-1}{n^n(n+1)};$$

i.e.,
$$\frac{1}{n^{n-1}} \sum_{k=0}^{n} \binom{n}{k} \frac{n^k}{k+1} = \left(1+\frac{1}{n}\right)^n - \frac{1}{n^n(n+1)}.$$

$$\therefore \qquad \lim_{n\to\infty} \frac{1}{n^{n-1}} \sum_{k=0}^{n} \binom{n}{k} \frac{n^k}{k+1} = e.$$

4.07 Suppose the ant never reaches the other end. Let the length of the rubber band after n^{th} stretching be b_n cm; then $b_n = 100 + 100n = 100(n+1)$. Let the ant be at the distance a_n cm, from the starting point, just after the stretching at the end of n minutes. After one more minute, it is at the distance $a_n + 10$, then by stretching, it is at the distance of a_{n+1}. Since the stretching is uniform, we have

$$\frac{a_{n+1}}{a_n+10} = \frac{b_{n+1}}{b_n} = \frac{n+2}{n+1} \Rightarrow \frac{a_{n+1}}{n+2} = \frac{a_n}{n+1} + \frac{10}{n+1}.$$

$$\therefore \qquad \sum_{k=0}^{n} \frac{a_{k+1}}{k+2} = \sum_{k=0}^{n} \left(\frac{a_k}{k+1} + \frac{10}{k+1}\right)$$

$$\Rightarrow \qquad \frac{a_{n+1}}{n+2} = 10\sum_{k=0}^{n} \frac{1}{k+1} = 10\sum_{k=1}^{n+1} \frac{1}{k}$$

$$\Rightarrow \qquad a_{n+1} = 10(n+2)\sum_{k=1}^{n+1} \frac{1}{k}.$$

Since $a_{n+1} < b_{n+1}$,

$$10(n+2)\sum_{k=1}^{n+1}\frac{1}{k} < 100(n+2) \Rightarrow \sum_{k=1}^{n+1}\frac{1}{k} < 10.$$

i.e., $1+\frac{1}{2}+\frac{1}{3}+\cdots$ is convergent. A contradiction. Therefore the ant reaches the other end of the band.

4.08 $f(1) = f(1\cdot 1) = f(1)+f(1) \Rightarrow f(1) = 0$. Since $f(\alpha^2) = 2f(\alpha)$, we can assume for our purpose $\alpha > 0$. Suppose for some $\ell \in \mathbb{R}^*$, $f(\ell) \neq 0$; then $f(\ell^2) = 2f(\ell) \Rightarrow f(\ell^2) \neq 0$. We can write $\ell^2 = \alpha^x$ for some $x \in \mathbb{R}$. Therefore $f(\alpha^x) \neq 0$. Since f is continuous, we can find a rational r such that $f(\alpha^r) \neq 0$. Writing $r = \frac{m}{n}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, $f(\alpha^m) = nf(\alpha^{m/n}) \Rightarrow f(\alpha^m) \neq 0$. Since $f(\alpha^m)+f(\alpha^{-m}) = f(\alpha^m\cdot\alpha^{-m}) = f(\alpha^0) = f(1) = 0$, $f(\alpha^{-m}) \neq 0$. Now $f(\alpha^{|m|}) = |m|f(\alpha) \Rightarrow f(\alpha) \neq 0$, a contradiction. Therefore $f \equiv 0$.

❺

Number Theory

5.01
$$2+\cfrac{1}{x+\cfrac{1}{y+\frac{1}{z}}}=\frac{37}{13}=2+\frac{11}{13}$$

$$\Rightarrow \quad x+\cfrac{1}{y+\frac{1}{z}}=\frac{13}{11}=1+\frac{2}{11}.$$

Since x is a positive integer, $x = 1$ and

$$y+\frac{1}{z}=\frac{11}{2}=5+\frac{1}{2}.$$

Since $\frac{1}{z} \leq 1$, we have $y = 5$ and therefore $z = 2$. Thus $(x, y, z) = (1, 5, 2)$.

5.02 By hypothesis

$$f\left(\frac{r}{s}\right)=a_0+a_1\left(\frac{r}{s}\right)+a_2\left(\frac{r}{s}\right)^2+\cdots+a_n\left(\frac{r}{s}\right)^n=0.$$

i.e., $a_0s^n + a_1rs^{n-1} + \cdots + a_{n-1}r^{n-1}s + a_nr^n = 0$.

Then $r \mid a_0s^n$ because all other terms of LHS are divisible by r. Since $(r, s) = 1$, $r \mid a_0$. Similar argument implies $s \mid a_n$.

Now to prove the second part, suppose r/s is a root of

$$1+x+x^{100}-x^{500}+x^{1000}=0 \qquad \cdots \ (**)$$

where $r, s \in \mathbb{Z}$ such that $(r, s) = 1$. Then by the first part, $r, s \mid 1$; i.e., $r, s \in \{-1, 1\}$; i.e., either -1 or 1 is a root. But neither $f(-1)$ nor $f(1)$ is 0. Hence $(**)$ can't have any rational root.

5.03 A contains at least 15 numbers $a_1, a_2, \ldots, a_{15}$ greater than 1. For each $i \le 15$, let p_i be a prime which divides a_i; let us assume that $p_1, p_2, \ldots, p_{15}$ are in the increasing order. Then $p_{14} \ge 43$, the 14^{th} prime and $p_{15} \ge 47$, the 15^{th} prime. Therefore $a_{14} \cdot a_{15} \ge p_{14} \cdot p_{15} \ge 43 \cdot 47 = 2021$.

5.04 If p is a prime such that $666 < p < 1000$ then the only numbers ≤ 2000, which are divisible by p are p and $2p$. Therefore the highest power of p which divides 2000! is p^2. Obviously the highest power of p dividing 1000! is p itself. Since $\binom{2000}{1000} = \frac{2000!}{(1000!)^2}$, p does not divide $\binom{2000}{1000}$. So let p be the largest prime < 666, viz., 661. It is obvious that the highest powers of p which divide 2000! and $(1000!)^2$ are p^3 and p^2 respectively; therefore p divides $\binom{2000}{1000}$.

5.05 Let a, b, c be three integers such that

$$a^3 + 5b^3 + 25c^3 - 15abc = 0. \qquad \cdots (1)$$

We shall show in two ways that

$$a = b = c = 0. \qquad \cdots (2)$$

Method 1 The algebraic identity

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x + y + z)\left[(x - y)^2 + (y - z)^2 + (z - x)^2\right]$$

implies

$$\text{either } a + 5^{1/3}b + 5^{2/3}c = 0 \quad \cdots (3)$$

$$\text{or } (a - 5^{1/3}b)^2 + (5^{1/3}b - 5^{2/3}c)^2 + (5^{2/3}c - a)^2 = 0 \quad \cdots (4)$$

Since $a, b, c \in \mathbb{Z}$, (3) is possible only when (2) holds.

(4) $\Rightarrow a = 5^{1/3}b = 5^{2/3}c$; again, this is possible only when (2) holds.

Method 2 Suppose (a, b, c) is a nonzero solution for the given equation. If a, b, c has any common divisor d, then $(a/d, b/d, c/d)$ is also a solution for the given equation. So we can assume that $\gcd(a, b, c) = 1$. 5 divides a^3 since it divides all other terms in (1). Therefore $5 \mid a$; i.e., $a = 5a_1$ for some $a_1 \in \mathbb{Z}$. Now (1) implies

$$25a_1^3 + b^3 + 5c^3 - 15a_1bc = 0.$$

Now by the same reasoning, $5 \mid b$; substituting $b = 5b_1$, we get

$$5a_1^3 + 25b_1^3 + c^3 - 15a_1b_1c = 0$$

which in turn implies that $5 \mid c$. Thus 5 divides all of a, b, c—a contradiction to our assumption. Therefore (2) holds.

5.06 We know that $\gcd(a, b)$ can written as $pa + qb$ for some $p, q \in \mathbb{Z}$. Since $\gcd(a, b) \mid c$, for some $\alpha \in \mathbb{Z}$, $c = \alpha \cdot \gcd(a, b) = \alpha pa + \alpha qb$. Now for any $t \in \mathbb{Z}$, $(\alpha p + bt, \alpha q - at)$ is a solution for the given equation.

5.07 Suppose x, y are two positive integers such that

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{20}}. \qquad \cdots (1)$$

Then $\dfrac{1}{\sqrt{y}} = \dfrac{1}{\sqrt{20}} - \dfrac{1}{\sqrt{x}} = \dfrac{\sqrt{x} - \sqrt{20}}{\sqrt{20} \cdot \sqrt{x}}$. Therefore

$$\frac{1}{y} = \frac{x + 20 - 4\sqrt{5x}}{20x}$$

implying that $\sqrt{5x}$ is rational. Now $x \in \mathbb{N} \Rightarrow \sqrt{5x} \in \mathbb{N}$. Hence $5x$ is the square of an integer which is divisible by

5. Therefore $5x = (5a)^2$ for some $a \in \mathbb{N}$. i.e., $x = 5a^2$. Similarly we can write $y = 5b^2$ for some $b \in \mathbb{N}$. Now (1)

$$\begin{aligned} &\Rightarrow \quad \frac{1}{a}+\frac{1}{b}=\frac{1}{2} \\ &\Rightarrow \quad 2(a+b)=ab \\ &\Rightarrow \quad (a-2)(b-2)=4 \\ &\Rightarrow \quad (a,b) \in \{(3,6),\ (4,4),\ (6,3)\}. \end{aligned}$$

Hence the solution-set is $\{(45,180),\ (80,80),\ (180,45)\}$.

5.08 Let $S = 1^5 + 2^5 + \cdots + 99^5$. For any $a, b \in \mathbb{Z}$, $a+b \mid a^5 + b^5$. Therefore 100 divides $\sum_{k=1}^{99}\left(k^5 + (100-k)^5\right)$; i.e., $100 \mid 2S$. $\therefore$ $10 \mid S$. So, the last digit of S is 0.

5.09 Let $x, y \in \mathbb{N}$ such that $3x + 5y = 1008$; then $3 \mid 5y$ $\Rightarrow$ $3 \mid y$ $\Rightarrow$ $y = 3k$, for some $k \in \mathbb{N}$. Now

$$\begin{aligned} 3x + 15k &= 1008 \\ \Rightarrow \quad x + 5k &= 336 \\ \Rightarrow \quad 5k &\le 335 \\ \Rightarrow \quad k &\le 67. \end{aligned}$$

Thus any solution pair is given by $(x,y) = (336-5k, 3k)$ where $1 \le k \le 67$ and therefore the number of solutions is 67.

5.10 Suppose $17^2 \mid n^2 - 3n - 19$. Since $n^2 - 3n - 19 = (n+7)(n-10)+51$, $17 \mid (n+7)(n-10)$; $\therefore$ $17^2 \mid (n+7)(n-10)$ ($\because$ $n+7 \equiv n-10 \pmod{17}$) $\Rightarrow$ $17^2 \mid (n^2 - 3n - 19) - (n+7)(n-10)$; i.e., $17^2 \mid 51$. A contradiction. Consequently $n^2 - 3n - 19$ is not a multiple of 289.

Second Method Suppose for some $k \in \mathbb{Z}$, $n^2 - 3n - 19 = 289k$. Then $n^2 - 3n - (19 + 289k) = 0$; therefore

$$\begin{aligned} n &= \frac{1}{2}\left(3 \pm \sqrt{9 + 4(19 + 289k)}\right) \\ &= \frac{1}{2}\left(3 \pm \sqrt{85 + 4 \cdot 289k}\right) \\ &= \frac{1}{2}\left(3 \pm \sqrt{17(5 + 68k)}\right). \end{aligned}$$

But $5+68k$ is not a multiple of 17 and hence $17(5+68k)$ cannot be a perfect square. Thus $289 \nmid n^2 - 3n - 19$.

5.11 Let

$$\begin{aligned} S &= \{a, b, c, d\} \\ X &= (a-b)(a-c)(a-d)(b-c)(b-d)(c-d) \\ P_k &= \{x \in S : x \equiv k \pmod 3\} \text{ for } k = 1, 2, 3 \text{ and} \\ Q_k &= \{x \in S : x \equiv k \pmod 2\} \text{ for } k = 1, 2. \end{aligned}$$

By Pigeon-hole principle, for some $k \leq 3$, P_k contains 2 elements of S; therefore 3 divides their difference and hence it also divides X. Next if either Q_1 or Q_2 contains three elements of S, say a, b, c, then $(a-b)(a-c)$ is divisible by 4; otherwise both Q_1 and Q_2 contain 2 elements each and in this case also we find $4 \mid X$. Hence X is divisible by 12.

Remark This problem is a particular case of the following result: If $x_1, x_2, \ldots, x_n$ are integers, then

$$\prod_{1 \leq i < j \leq n} (i - j) \text{ divides } \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

5.12 Choose m such that $2, n \mid m + 1$. (For example, we can have $m = 2n - 1$.) Note that every term of the sequence is even. Let u_k denote the k^{th} term of the sequence. Then $u_{k+1} = m^{u_k - 1} + 1 \equiv (-1)^{u_k - 1} + 1 \pmod n = -1 + 1 = 0$. Thus n divides every term of the sequence.

5.13 Let s be the sum of the ages of the ten persons. Since two persons have written the same sum, say γ, we have

$$\gamma + 82 + 83 + 84 + 85 + 87 + 89 + 90 + 91 + 92 = 9s.$$

i.e., $\gamma + 783 = 9s$, hence $9 \mid \gamma$. Thus $\gamma = 90$ and $9s = 873$ $\Rightarrow s = 97$. Therefore the ages of the persons are given by $97 - x$ where $x = 82, 83, 84, 85, 87, 89, 90, 90, 91, 92$; i.e., 15,14,13,12,10,8,7,7,6,5.

5.14 Let $(a_i)_{i=1}^{16}$ be one such sequence. For any $k \leq 9$, by using (b), we have $a_k + a_{k+1} + \cdots + a_{k+6} = -1 = a_{k+1} + \cdots + a_{k+7}$; $\therefore\ a_k = a_{k+7}$. Similarly, by using (c), we have $a_k = a_{k+11}$, for all $k \leq 5$. Thus by (a), (b) and (c) we have the following data: For all $i, j \leq 16$,

$$\text{if } i + j = 17 \text{ or } |i - j| = 7 \text{ or } 11, \text{ then } a_i = a_j.$$

Using this we have $a_1 = a_8 = a_{15} = a_4 = a_{11} = a_6 = a_{13} = a_2 = a_9 = a_{16} = a_5 = a_{12}$ and $a_3 = a_{10} = a_7 = a_{14}$. Now using (b) and (c) for the first 7 and 11 terms respectively, we get $5a_1 + 2a_3 = -1$ and $8a_1 + 3a_3 = 1$; solving, we have $a_1 = 5$ and $a_3 = -13$. Thus the sequence which satisfies our requirement is unique, viz.,

$$5,\ 5,\ -13,\ 5,\ 5,\ 5,\ -13,\ 5,\ 5,\ -13,\ 5,\ 5,\ 5,\ -13,\ 5,\ 5.$$

5.15 Suppose $a - d$, a, $a + d$ are three integers such that $(a - d)^2 + a^2 + (a + d)^2 = 1994$. Then $3a^2 + 2d^2 = 1994$. So, a is even and putting $a = 2b$, we get $6b^2 + d^2 = 997 \equiv 5 \pmod 8$. Now d being odd, $d^2 \equiv 1 \pmod 8$ and $6b^2 \equiv 6$ or 0 (depending upon whether b is odd or even). Therefore $d^2 + 6b^2 \equiv 7$ or $1 \pmod 8$. A contradiction. Hence there can't be such a triplet.

5.16 $2 \mid y^2 \Rightarrow 4 \mid 2xz \Rightarrow 2 \mid x$ or $z \Rightarrow 2 \mid x$ and z ($\because\ 2 \mid x+z$). Let $x = 2x_1$, $y = 2y_1$ and $z = 2z_1$. Then $2x_1z_1 = y_1^2$ and $x_1 + z_1 = 997$. Again, y_1 and one of x_1, z_1, say x_1, are even. Writing $x_1 = 2x_2$ and $y_1 = 2y_2$, we have $x_2z_1 = y_2^2$ and $2x_2 + z_1 = 997$. Since 997 is a prime, x_2 and z_1 are relatively prime; therefore each is a square, since their product is a square. Since any square is of the form $8n$, $8n+1$ or $8n+4$, $2x_2 \equiv 0$ or $2 \pmod 8$ and $z_1 \equiv 1 \pmod 8$ ($\because\ z_1$ is odd). Hence $2x_2 + z_1 \equiv 1$ or $3 \pmod 8$; a contradiction because $997 \equiv 5 \pmod 8$.

5.17 Let
$$S = \{n \in \mathbb{N} : n < 175\},$$
$$A = \{n \in S : 5 \mid n\}$$
and
$$B = \{n \in S : 7 \mid n\}.$$
Then
$$A \cap B = \{n \in S : 35 \mid n\}.$$

Now the required number is

$$\begin{aligned} |S| - |A \cup B| &= |S| - (|A| + |B| - |A \cap B|) \\ &= 174 - \left\lfloor \frac{174}{5} \right\rfloor - \left\lfloor \frac{174}{7} \right\rfloor + \left\lfloor \frac{174}{35} \right\rfloor \\ &= 174 - 34 - 24 + 4 = 120. \end{aligned}$$

5.18 Since $(n-1)n(n+1) \equiv 0 \pmod 3$ and by hypothesis, $n(n+1) \equiv -1 \pmod 3$, we have $(n-1)(-1) \equiv 0 \pmod 3$; i.e., $3 \mid n-1$.

Alternative Solution Since 3 divides $n(n+1)+1$, it does not divide $n(n+1)$; therefore $3 \mid n-1$ ($\because\ 3 \mid (n-1)n(n+1)$).

5.19 Since $n^4 - n^2 + 64 > n^4 - 2n^2 + 1 = (n^2-1)^2$, for some non-negative integer k, $(n^4 - n^2 + 64) = (n^2 + k)^2 =$

$n^4+2n^2k+k^2$; i.e., $n^2 = \dfrac{64-k^2}{2k+1}$ from which we find that the possible values 64,1,0 for n^2 are obtained when $k = 0, 7, 8$ respectively. Hence $n \in \{0, \pm 1, \pm 8\}$.

5.20 Let $\lambda = 11 + 11\sqrt{11a^2+1}$.

Then $(\lambda - 11)^2 = 11^2(11a^2+1)$;

simplifying we get $\lambda(\lambda - 22) = 11^3a^2$.

Putting $|a| = \dfrac{r}{s}$ where $r, s \in \mathbb{N}$ such that $(r, s) = 1$, gives $\lambda(\lambda-22)s^2 = 11^3r^2$. Since $11^2 \nmid s$ because otherwise 11 would divide r, $11 \mid \lambda$. Writing $\lambda = 11\mu$, we get $\mu(\mu-2)s^2 = 11r^2$. Since $11 \nmid s$ for otherwise we would have $11 \mid r$, it follows that $s = 1$. Thus we have $\mu(\mu-2) = 11r^2$. Since $\mu - 2$ and μ are consecutive odd integers, they are relatively prime. If $11 \mid \mu - 2$, then μ is a square of the form $11n+2$ which is not possible. Therefore $11 \mid \mu$ and hence $\mu = 11n^2$ for some $n \in \mathbb{N}$. Thus we have $\lambda = 11\mu = 11^2n^2$.

5.21 Let p be a prime number such that $\dfrac{2^{p-1}-1}{p}$ is a square. Writing $(2^{p-1}-1)$ as $(2^{(p-1)/2}-1)(2^{(p-1)/2}+1)$, we have two possibilities.

Case (i) $p \mid 2^{(p-1)/2} - 1$.

Then $2^{(p-1)/2}+1$ which is relatively prime to $2^{(p-1)/2}-1$ is a square, say a^2. Thus $2^{(p-1)/2} = a^2 - 1 = (a-1)(a+1)$. Let $(a-1) = 2^r$; then $a+1 = 2(2^{r-1}+1)$ is a power of 2 $\Rightarrow r = 1$. Therefore $2^{(p-1)/2} = 2^3 \Rightarrow p = 7$.

Case (ii) $p \mid 2^{(p-1)/2} + 1$.

Now $2^{(p-1)/2}-1$ is a square. Since a square is not congruent to -1 modulo 4, $4 \nmid 2^{(p-1)/2}$; therefore $\frac{p-1}{2} = 1$. i.e., $p = 3$.

Thus the desired primes are 3 and 7.

5.22 $5 \mid a_1a_2a_3a_4a_5 \Rightarrow a_5 = 5$. Since the number $a_1 \cdots a_k$ is divisible by k it must be even when $k = 2$, 4 or 6. Hence $\{a_2, a_4, a_6\} = \{2, 4, 6\}$. Therefore $\{a_1, a_3\} = \{1, 3\}$. Now $3 \mid a_1a_2a_3 \Rightarrow 3 \mid a_1 + a_2 + a_3 \Rightarrow 3 \mid 4 + a_2 \Rightarrow a_2 = 2$. Also $4 \mid a_1a_2a_3a_4 \Rightarrow 4 \mid a_3a_4 \Rightarrow a_4 = 6$. Consequently $a_6 = 4$. Thus the required numbers are 123654 and 321654.

5.23 For any $k, n \in \mathbb{N}$, the highest power of k which divides n is given by

$$\psi_k(n) = \left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{n}{k^2} \right\rfloor + \left\lfloor \frac{n}{k^3} \right\rfloor + \cdots.$$

The number of zeros with which n ends when written in decimal system is $\min\{\psi_2(n), \psi_5(n)\} = \psi_5(n)$. Hence we have to find n such that $\psi_5(n) = 1993$. Since

$$\psi_5(n) \leq \frac{n}{5} + \frac{n}{5^2} + \cdots = \frac{n}{5}\left(1 + \frac{1}{5} + \cdots\right) = \frac{n}{5} \cdot \frac{5}{4} = \frac{n}{4},$$

we have $\frac{n}{4} \geq 1993 \Rightarrow n > 7970 \Rightarrow n \geq 7975$. We find that $\psi_5(7975) = 1991$. Therefore $\psi_5(7985) = \psi_5(7975) + 2 = 1993$. Hence 7985 is one solution; other numbers are 7986, 7987, 7988 and 7989.

5.24 Let

$$(x+1) + (x+2) + \cdots + (x+k) = 3^{11}.$$

Then $\frac{k}{2}(2x+k+1) = 3^{11}$; i.e., $k(2x+k+1) = 2 \cdot 3^{11}$. Let $k = 2^{\alpha}3^{\beta}$. Then $k^2 < 2 \cdot 3^{11} \Rightarrow 2^{2\alpha}3^{2\beta} < 2 \cdot 3^{11} \Rightarrow 3^{2\beta} < 2 \cdot 3^{11} < 3^{12} \Rightarrow 2\beta < 12 \Rightarrow \beta \leq 5$. Since $\alpha \leq 1$, we have $k \leq 2 \cdot 3^5 = 486$; k takes this value when $x = 121$; therefore 486 is the largest possible value of k.

5.25 Since a, b, c, d, e are consecutive integers,

$$b + c + d = 3c$$
$$\text{and} \quad a + b + c + d + e = 5c.$$

Now $3 \mid 3c \Rightarrow 3^2 \mid 3c$ ($\because$ $3c$ is a square) $\Rightarrow 3 \mid c \Rightarrow 3 \mid 5c \Rightarrow 3^3 \mid 5c$ ($\because$ $5c$ is a cube) $\Rightarrow 3^3 \mid c$. Also $5 \mid 5c \Rightarrow 5^3 \mid 5c \Rightarrow 5^2 \mid c$. Therefore $3^3 5^2 \mid c$; i.e., $675 \mid c$. Thus, 675—being a possible value of c—is the smallest of such numbers.

5.26 Let x be a non-negative integer and n, a positive integer such that

$$(x+1) + (x+2) + \cdots + (x+n) = 1000.$$

Then $\frac{n}{2}(2x+1+n) = 1000$; i.e., $n(2x+1+n) = 2^4 \times 5^3$. Let $n = 2^a 5^b$; then $2x + 1 + 2^a 5^b = 2^{4-a}5^{3-b}$; i.e., $2x+1 = 2^{4-a}5^{3-b} - 2^a 5^b$. Since $2x+1$ is odd, a is either 0 or 4.

Case (a) $a = 0$.

Then $2x+1 = 2^4 5^{3-b} - 5^b$. Now $b = 0 \Rightarrow n = 1$ and $x = 999$; $b = 1 \Rightarrow n = 5$ and $x = 197$; $b = 2 \Rightarrow n = 25$ and $x = 27$; $2x+1$ being positive, $b \neq 3$.

Case (b) $a = 4$.

Then $2x+1 = 5^{3-b} - 2^4 5^b$; again $2x+1$ being positive, $b = 0$ which implies $n = 16$ and $x = 54$.

Hence the required sets are $\{1000\}$, $\{198, 199, \ldots, 202\}$, $\{28, 29, \ldots, 52\}$, $\{55, 56, \ldots, 70\}$.

5.27 Let the year of birth be y. It is given that $1800 \leq y < 1850$ and $x^2 = y + x$. $\therefore 1800 \leq (x-1)x < 1850$. A simple check gives $x = 43$ and hence $y = 42 \times 43 = 1806$.

5.28 $(abcd) = (abc) \times (bd) \Rightarrow (abc)10 + d = (abc)10b + (abc)d \Rightarrow b = 1$ ($\because$ abc and the leading digit of bd, viz., b are positive). Note that we also have $d = 0$.

5.29 Let $2^m + 3^n = k^2$. Since $(-1)^m \equiv 2^m \equiv k^2 \equiv 1 \pmod 3$ ($\because$ $3 \nmid k$), m is even, say $2p$. Now $(k - 2^p)(k + 2^p) = 3^n$ $\Rightarrow$ $k - 2^p = 1$ and $k + 2^p = 3^n$ $\Rightarrow$ $2^{p+1} + 1 = 3^n$. Since $(-1)^n \equiv 3^n \pmod 4 = 2^{p+1} + 1 \equiv 1$, n is even, say $2q$. Now $(3^q - 1)(3^q + 1) = 2^{p+1} \Rightarrow 3^q - 1 = 2 \Rightarrow 3^q = 3 \Rightarrow q = 1$ and hence $p = 2$. So we have only one solution, viz., $(4, 2)$.

5.30 For any $m \in \mathbb{Z}$, let $\theta(m)$ denote the highest power of 3 which divides m. Let $t_n = 2^{3^n} + 1$ where $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$,

$$t_{n+1} = \left(2^{3^n}\right)^3 + 1 = (t_n - 1)^3 + 1$$

which by simplification gives $t_{n+1} = t_n^3 - 3t_n^2 + 3t_n$. Now let us show by induction $\theta(t_n) = 3^{n+1}$. When $n = 1$, it is obvious. Assume $n \geq 1$ such that $\theta(t_n) = 3^{n+1}$. Then

$$\begin{aligned} \theta(t_{n+1}) &= \theta(t_n^3 - 3t_n^2 + 3t_n) \\ &= \theta(t_n)\theta(t_n^2 - 3t_n + 3) \\ &= 3^{n+1} \cdot 3 = 3^{n+2}. \end{aligned}$$

5.31 Let x and y be two numbers such that $3 \mid x^2 + y^2$. We can write $x^2 \equiv a \pmod 3$ and $y^2 \equiv b \pmod 3$ where

$a, b \in \{0, 1\}$. Therefore $a + b \equiv x^2 + y^2 \equiv 0 \pmod 3 \Rightarrow a = b = 0 \Rightarrow x \equiv y \equiv 0 \pmod 3$.

5.32

$$\begin{aligned} & n^2 + 3n + 2 \equiv 0 \pmod 6 \\ \Longleftrightarrow\quad & (n+1)(n+2) \equiv 0 \pmod 2 \text{ and} \\ & (n+1)(n+2) \equiv 0 \pmod 3 \\ \Longleftrightarrow\quad & (n+1)(n+2) \equiv 0 \pmod 3 \\ & (\text{Note: } (n+1)(n+2) \text{ is even for all } n) \\ \Longleftrightarrow\quad & 3 \nmid n \end{aligned}$$

Therefore the required number is

$$1991 - \left\lfloor \frac{1991}{3} \right\rfloor = 1991 - 663 = 1328.$$

5.33 Let as prove the following general result: If m is an even positive integer and n is an odd positive integer, then

$$S = 1^n + 2^n + \cdots + m^n$$

is divisible by $m + 1$.

Since n is odd, for any $a, b \in \mathbb{N}$, $a + b \mid a^n + b^n$. Therefore, for $k = 1, \ldots, m$,

$$k + m + 1 - k \mid k^n + (m + 1 - k)^n;$$

i.e., $m + 1 \mid k^n + (m + 1 - k)^n$.

Summing up these m relations, we get $m+1 \mid 2S$. Since $m + 1$ is odd, $m + 1 \mid S$.

5.34 For any $n \in \mathbb{N}$, let

$$u_n = \begin{cases} 3 & \text{if } n \equiv 1 \pmod 4 \\ 9 & \text{if } n \equiv 2 \pmod 4 \\ 7 & \text{if } n \equiv 3 \pmod 4 \\ 1 & \text{if } n \equiv 4 \pmod 4). \end{cases}$$

Note that $3u_n \equiv u_{n+1} \pmod{20}$. Since $20 \mid 3 - u_1$ and for any $n \in \mathbb{N}$,

$$20 \mid 3^n - u_n \Rightarrow 20 \mid 3^{n+1} - 3u_n \Rightarrow 20 \mid 3^{n+1} - u_{n+1}$$

it follows by induction, $20 \mid 3^n - u_n$ for all $n \in \mathbb{N}$. Hence ten's digit of 3^n is even. (Note that unit's digit of 3^n is u_n.)

5.35 In the binomial expansion of $\left(1 + \frac{6}{13}\right)^{15}$, the third term $\frac{15 \cdot 14}{1 \cdot 2}\left(\frac{6}{13}\right)^2 > 13$. Therefore

$$\begin{aligned} &\left(1 + \frac{6}{13}\right)^{15} > 13 \\ \Rightarrow\quad &\left(1 + \frac{6}{13}\right)^{31} > 13^2 \\ \Rightarrow\quad &19^{31} > 13^{33} \\ \Rightarrow\quad &19^{93} > 13^{99}. \end{aligned}$$

In the binomial expansion of $(1 + 18)^{93}$ the first term is 1; the second term is $93 \cdot 18 \equiv 12 \cdot 18 \pmod{81}$; all other terms are divisible by 81. In the expansion of $(1 + 12)^{99}$ the first term is 1; the second term is $99 \cdot 12 \equiv 18 \cdot 12 \pmod{81}$; the next two terms $\frac{99 \cdot 98}{1 \cdot 2} \cdot 12^2$, $\frac{99 \cdot 98 \cdot 97}{1 \cdot 2 \cdot 3} 12^3$ are divisible by 81; the other terms being multiples of 12^4 are also divisible by 81. Hence we have $19^{93} \equiv 13^{99} \pmod{81}$. $\therefore\ 162 \mid 19^{93} - 13^{99}$ ($\because\ 19^{93} - 13^{99}$ is even).

5.36 $19^{88} - 1 = (1 + 18)^{88} - 1 = (1 - 20)^{88} - 1.$

In the binomial expansion of $(1+18)^{88}$, the first term is 1; the second is 18; all other terms are divisible by 18^2; therefore the maximum power of 3 which divides $(1+18)^{88}-1$ is 3^2.

In the expansion of $(1-20)^{88}$, the first term is 1; the second is $-20\cdot 88 = -2^5\cdot 55$; the third one is $20^2\times\dfrac{88\cdot 87}{2} = 2^6\cdot 5^2\cdot 11\cdot 87$ and therefore is divisible by 2^6; all other terms are obviously divisible by 2^6. Hence the maximum power of 2 which divides $(1-20)^{88}-1$ is 2^5.

Hence the sum of the factors of $19^{88}-1$ which are of the form $2^a\cdot 3^b$ where $a, b > 0$ is

$$(2+2^2+2^3+2^4+2^5)(3+3^2) = 744.$$

5.37 Let x be any positive integer; let $y = 2x+1$. Choose $a = x^2 - y^2$. (Note that different x's yield different a's.) Now $a(y-2x) = x^2 - y^2 \;\Rightarrow\; x^2+2ax = y^2+ay$. Therefore putting $b = -(x^2+2ax) \;\Rightarrow\; x^2+2ax+b = y^2+ay+b = 0$. Note that $(a,b) = 1$ because for any prime p, $p \mid a, b \Rightarrow p \mid x^2, y^2 \Rightarrow p \mid x, y$—a contradiction.

5.38 $(x+y)^2 < 1994 < 45^2 \quad\Rightarrow\quad (x+y) \le 44$ and

$$\begin{aligned}(x+y)(x+y+3) &= (x+y)^2+3x+3y\\ &> (x+y)^2+3x+y = 1994 > 43\cdot 46\\ \Rightarrow\qquad x+y &\ge 44.\end{aligned}$$

Therefore $x+y = 44$. Now

$$(x+y)^2+3(x+y)-2y = 1994 \Rightarrow y = 37.$$

5.39 Let n be the number of the pages of the book. Suppose the leaf torn bears the page numbers k and $k+1$. Then

$$\frac{n(n+1)}{2} = 15000 + k + (k+1).$$

Therefore

$$n(n+1) = 30000 + 4k + 2. \qquad \cdots (**)$$

Since $k \geq 1$, we have

$$n(n+1) \geq 30006 > 172 \cdot 173 \Rightarrow n \geq 173.$$

Since $k \leq n$,

$$\begin{aligned} & n(n+1) \leq 30000 + 4n + 2; \\ \text{i.e.,} \quad & n^2 - 3n + 2 \leq 30004 \\ \Rightarrow \quad & (n-2)(n-1) < 173 \cdot 174 \\ \Rightarrow \quad & n - 2 < 173 \\ \Rightarrow \quad & n \leq 174. \end{aligned}$$

Note that $n \neq 174$, for otherwise substituting $n = 174$ in $(**)$, we would get $k = 112$, a contradiction because k being a right-page number, should be odd. So taking $n = 173$, we get from $(**)$, that $k = 25$; hence the page numbers of the torn leaf are 25 and 26.

5.40 Since 19 and 92 are relatively prime, by (an) Euler's theorem, $19^{\varphi(92)} \equiv 1 \pmod{92}$. Since $92 = 2^2 \cdot 23$,

$$\varphi(92) = 92\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{23}\right) = 44.$$

$\therefore\ 19^{44} \equiv 1 \Rightarrow 19^{88} \equiv 1 \Rightarrow 19^{92} \equiv 19^4 = (361)^2 \equiv (-7)^2 = 49$. therefore the remainder is 49.

5.41 $3^5 = 243 \equiv 1 \pmod{11}$. Therefore $3^{1994} = 3^4 \cdot 3^{1990} \equiv 81 \cdot 1 \equiv 4$. Therefore the remainder, when $3^{1994} + 2$ is divided by 11 is 6.

5.42 $1001 = 7 \cdot 11 \cdot 13$; therefore every term of $\sum_{n=14}^{100} n!$ is divisible by 1001; the highest power of 5 which divides the first term is 5^2 $\left(\because \left\lfloor \frac{14}{5} \right\rfloor + \left\lfloor \frac{14}{5^2} \right\rfloor + \cdots = 2\right)$; therefore 1000 does not divide 14! but every other term divisible by 1000. Therefore $\sum_{n=14}^{100} n!$ is not divisible by 1000.

5.43 Since $25^2 = 625$ and $25^3 = 15625 \equiv 625 \pmod{1000}$, we find successively, $25^2 \equiv 25^3 \equiv 25^4 \equiv \cdots \equiv 25^{63} \pmod{1000}$; therefore $25^{63} \equiv 625 \pmod{1000}$. Hence $25^{63} = 1000m + 625$.

Next $63 \equiv -1 \pmod 8 \Rightarrow 63^{25} \equiv (-1)^{25} = -1 \Rightarrow 63^{25} = 8k - 1$.

$\therefore$ $25^{63} \times 63^{25} = (1000m + 625)(8k - 1) \equiv -625 \equiv 375 \pmod{1000}$. Hence the last three digits are 3,7,5.

5.44 From the relation $a_{n+1} = 3a_n + 1$, for all $n \geq 0$, it follows that if $a_n \equiv 0 \pmod{11}$, then $a_{n+1} \equiv 1 \Rightarrow a_{n+2} \equiv 4 \Rightarrow a_{n+3} \equiv 2 \Rightarrow a_{n+4} \equiv 7 \Rightarrow a_{n+5} \equiv 0$. Since $a_0 = 0$, we have for all $n \geq 0$, $a_{5n} \equiv 0$ and $a_{5n+4} \equiv 7$. Hence a_{1995} is divisible by 11; but a_{1994} is not divisible by 11.

5.45 Suppose $a \in \mathbb{Z}$ such that $a^2 = (n+1)^3 - n^3$. Then

$$a^2 = 3n(n+1) + 1. \qquad \cdots (1)$$

Now $4a^2 - 1 = 12n(n+1) + 3 = 3(4n^2 + 4n + 1)$

$\Rightarrow$ $(2a-1)(2a+1) = 3(2n+1)^2$.

Therefore $3 \mid 2a - 1$ or $2a + 1$.

If $3 \mid 2a - 1$, then $2a - 1 = 3m$ for some $m \in \mathbb{Z}$; then

$$3m(2a+1) = 3(2n+1)^2 \Rightarrow m(2a+1) = (2n+1)^2.$$

Since m and $2a+1$ are relatively prime $[\because\ (2a+1, 2a-1) = 1]$, $2a+1$ is an odd square, say $(2r+1)^2$. Then $a = 2r^2 + 2r$ is even; a contradiction because by (1), a^2 is odd. Therefore $3 \mid 2a+1$. Now proceeding as above, we see that $2a - 1 = (2r+1)^2$, for some $r \in \mathbb{Z}$; i.e., $a = r^2 + (r+1)^2$.

5.46 Consider the remainders obtained when the given 181 squares are divided by 19. If n is any integer, then n is congruent modulo 19 to one of the numbers $0, 1, 2, \ldots, 17$ or 18 and hence n^2 is congruent to $0, 1, 4, 9, 16, 6, 17, 11, 7$ or 5. Hence there are only 10 possible values for the remainders. Since there are 181 remainders, one of them must appear at least 19 times. Choose 19 numbers congruent to that remainder.

5.47 Let $n^2 + 19n + 92 = m^2$, where m is a non-negative integer. Then $n^2 + 19n + 92 - m^2 = 0$. Solving for n, we get

$$n = \frac{1}{2}\left(-19 \pm \sqrt{4m^2 - 7}\right).$$

Therefore $4m^2 - 7$ is a square; i.e., $4m^2 - 7 = p^2$ where $p \in \mathbb{N}$. $\therefore$ $(2m - p)(2m + p) = 7$. Therefore $2m + p$ being positive, is 7 and $2m - p = 1$. Hence $4m = 8 \Rightarrow m = 2$. Thus we have

$$\begin{aligned} & n^2 + 19n + 92 = 4 \\ \Rightarrow\ & n^2 + 19n + 88 = 0 \\ \Rightarrow\ & (n+8)(n+11) = 0 \\ \Rightarrow\ & n = -8 \text{ or } -11. \end{aligned}$$

Alternative Solution Let $n^2 + 19n + 92 = m^2$, where m is a non-negative integer. Then

$$m^2 - 4 = n^2 + 19n + 88$$
$$\Rightarrow \quad (m-2)(m+2) = (n+8)(n+11).$$

Note that $m \neq 0, 1$ because neither -4 nor -3 can be written as a product of two numbers which differ by 3. Claim: $m = 2$. Suppose the contrary. Then $(n+8)(n+11)$ is positive. Therefore one of $n+8, -(n+11)$ is positive; Let it be k. Then $(m-2)(m+2) = k(k+3) \Rightarrow m - 2 < k$ and $k + 3 < m + 2 \quad \Rightarrow \quad m < k + 2 < m + 1$; a contradiction. Therefore the claim holds and the rest is obvious.

5.48 $2137^{753} = (2130 + 7)^{753} \equiv 7^{753} \pmod{10} = 7 \cdot 7^{752} = 7 \cdot (7^2)^{376} = 7 \cdot (49)^{376} \equiv 7(-1)^{376} = 7$. Therefore the last digit of 2137^{753} is 7.

5.49 Suppose the existence of $x, y \in \mathbb{Z}$ such that $3x^{10} - y^{10} = 1991$. Note that $11 \mid 1991$. Therefore neither x nor y is divisible by 11 for otherwise 11 would divide both $\Rightarrow$ $11^{10} \mid 3x^{10} - y^{10} \Rightarrow 11^{10} \mid 1991$, an impossibility. Hence x and y are prime to 11. $\therefore\ x^{10} \equiv y^{10} \equiv 1 \pmod{11} \Rightarrow$ $1991 = 3x^{10} - y^{10} \equiv 3 - 1 = 2$. A contradiction.

5.50
$$\begin{aligned} 2^{200} - 2^{192} \cdot 31 + 2^n &= 2^{192}(2^8 - 31) + 2^n \\ &= 2^{192}(256 - 31) + 2^n \\ &= 2^{192} \cdot 225 + 2^n. \end{aligned}$$

Therefore for some $m \in \mathbb{N}$,

$$\begin{aligned} 2^n &= m^2 - 2^{192} \cdot 225 \\ &= m^2 - (2^{96} \cdot 15)^2 \\ &= (m - 2^{96} \cdot 15)(m + 2^{96} \cdot 15). \end{aligned}$$

So $m - 2^{96} \cdot 15 = 2^\alpha$ and $m + 2^{96} \cdot 15 = 2^{\alpha+\beta}$ for some non-negative integers α, β. Hence $2^{97} \cdot 15 = 2^{\alpha+\beta} - 2^\alpha =$

$2^{\alpha}(2^{\beta}-1) \Rightarrow 2^{\alpha} = 2^{97}$ and $2^{\beta}-1 = 15$. i.e., $\alpha = 97$ and $\beta = 4$. $\therefore\ n = 2\alpha + \beta = 198$.

5.51 $(56789)^{41} \equiv 89^{41} \pmod{100} \equiv 89(1-90)^{40} \equiv 89$ ($\because$ in the binomial expansion of $(1-90)^{40}$ all the terms except the first are divisible by 100).

5.52 Let n be any integer; when it is odd,

$$n^4 - 1 = (n-1)(n+1)(n^2+1)$$

is divisible by 16 because $n-1, n+1, n^2+1$ are all even and $n-1, n+1$ being consecutive even integers, one of them is divisible by 4. When n is even $n^4 \equiv 0 \pmod{16}$. Thus $n^4 \equiv 0$ or 1. Now for any $x, y, z, t \in \mathbb{Z}$,

$$x^4 + y^4 + z^4 - t^4 \equiv \alpha$$

where $\alpha \in \{-1, 0, 1, 2, 3, \}$. Since $1991 \equiv 7$,

$$x^4 + y^4 + z^4 - t^4 \not\equiv 1991.$$

5.53 Let $S = \{1, 4, 7, \ldots, 100\}$, $S_1 = \{4, 100\}$, $S_2 = \{7, 97\}$, …, $S_{16} = \{49, 55\}$, $S_{17} = \{1\}$, $S_{18} = \{52\}$.

Then $\{S_i : i \leq 18\}$ is a partition of S. Clearly, any subset of S having 19 elements must contain two elements from some S_i. Sum of these two elements is 104. A subset of S containing 18 elements such that the sum of no two distinct integers from the set equals 104 can be chosen by picking one element from each S_i, $1 \leq i \leq 18$. For example, $\{1, 4, 7, 10, \ldots, 49, 52\}$ is such a set.

5.54 $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{n} \Rightarrow x, y > n$. Therefore $x = n + a$ and $y = n + b$ where $a, b \in \mathbb{N}$. Then

$$
\begin{aligned}
& \frac{1}{n+a}+\frac{1}{n+b}=\frac{1}{n} \\
\Rightarrow\quad & (n+b+n+a)n=(n+a)(n+b) \\
\Rightarrow\quad & n^2=ab.
\end{aligned}
$$

Therefore $s(n)$ is the number of divisors of n^2. Let $n=p_1^{\alpha_1}\cdots p_m^{\alpha_m}$ be the prime-factorization of n where $\alpha_1\geq\cdots\geq\alpha_m$. Then

$$s(n)=(1+2\alpha_1)\cdots(1+2\alpha_m).$$

Therefore $s(n)=5\Rightarrow 1+2\alpha_1=5$ and $m=1$. $\therefore$ $n=p_1^2$. Thus the required set is $\{p^2 : p \text{ is a prime}\}$.

5.55 Let p_1 and p_2 be two consecutive odd primes. Since p_1+p_2 is even, $\frac{1}{2}(p_1+p_2)\in\mathbb{Z}$. Further $\frac{1}{2}(p_1+p_2)$ being in between the consecutive primes p_1 and p_2, is a composite number. Therefore $p_1+p_2=2\cdot\frac{p_1+p_2}{2}$ has at least three prime factors.

5.56 if any one of a,b,c is divisible by p, then the conclusion holds. So assume that $p\nmid a,b,c$. Now by Fermat's theorem $a^{p-1}\equiv 1 \pmod{p}\Rightarrow a^{2n}\equiv 1\pmod{p}\Rightarrow a^n\equiv\pm1$. Similarly $b^n\equiv\pm1$ and $c^n\equiv\pm1$. Therefore two of a^n,b^n,c^n are congruent modulo p and hence their difference is divisible by p. This implies that

$$p\mid abc(a^n-b^n)(b^n-c^n)(c^n-a^n).$$

5.57

$$
\begin{aligned}
& n=810(0.d25d25d25\cdots) \\
\Rightarrow\quad & 1000n=810(d25.d25d25\cdots) \\
\Rightarrow\quad & 999n=810\times d25 \\
\Rightarrow\quad & 37n=30\times d25 \\
\Rightarrow\quad & 37\mid d25
\end{aligned}
$$

which is possible only when $d = 9$. Therefore

$$n = \frac{30 \times 925}{37} = 750.$$

5.58 Let $\theta, n \in \mathbb{N}$ and $n \geq 3$. We know that the exterior angle of the regular polygon with n sides is θ° if and only if $n \cdot \theta = 360$. Hence what we need is the least positive integer which does not divide 360; that is obviously 7.

5.59 When $p = 3$, $8p + 1 = 25$, a composite number. Otherwise $3 \nmid (8p-1)8p$. $\Rightarrow 3 \mid 8p+1$ $[\because\ 3 \mid (8p-1)8p(8p+1)]$. Hence $8p+1$ is not a prime.

5.60 Since

$$\begin{aligned} &(xy-1)^2 - (x+1)^2 - (y+1)^2 \\ &= (xy-1-x-1)(xy+x) - (y+1)^2 \\ &= (y+1)(x^2y - x^2 - 2x - y - 1) \\ &= (y+1)\left[y(x^2-1) - (x+1)^2\right] \\ &= (x+1)(y+1)(xy - y - x - 1) \\ &= (x+1)(y+1)[(x-1)(y-1) - 2] \end{aligned}$$

and $(x-1)(y-1) = 2 \iff (x,y) \in \{(2,3), (3,2), (-1,0), (0,-1)\}$, the solution-set is

$$\{(x,-1),\ (-1,y),\ (3,2),\ (2,3)\ :\ x, y \in \mathbb{Z}\}.$$

5.61 By the hypothesis $\dfrac{a+b}{ab} = \dfrac{1}{c}$; i.e., $(a+b)c = ab$. Let p be any prime which divides $a+b$; then p divides one of a, b and therefore both. Since $\gcd(a,b,c) = 1$, p does not divide c; therefore for any $k \in \mathbb{N}$, $p^k \mid a \iff p^k \mid b$; hence the maximum power of p which divides $a+b$ is the square of the maximum power of p which divides a. This implies that $a+b$ is a square.

5.62 Since $p(x) = p(-x)$ for any real x, we can assume that a and b are non-negative and also that $a > b$. Since $p(a) - p(b) = (a-b)(a+b)$, if $a - b > 1$, then $p(a) - p(b)$ is composite; So assume that $a - b = 1$. Now

$$p(a) - p(b) = (b+1) + b = 2b + 1$$

$$\text{and}\quad \begin{aligned} p(a) + p(b) &= (b+1)^2 + 40 + b^2 + 40 \\ &= 2b^2 + 2b + 81 \\ &= 2b(b+1) + 81. \end{aligned}$$

If $3 \mid b(b+1)$ then $p(a) + p(b)$ is composite; otherwise $3 \mid b - 1$ and hence $3 \mid 2(b-1) + 3$; i.e., $3 \mid p(a) - p(b)$; therefore $p(a) - p(b)$ is composite, if $b \neq 1$; when $b = 1$, $p(a) + p(b) = 85$ and so it is composite.

5.63 We can write $f(x) = (x-k)g(x) + f(k)$ where $g(x)$ is a polynomial with integral coefficients. Then

$$\begin{aligned} f(1) &= (1-k)g(1) + f(k) \\ \text{and}\quad f(2) &= (2-k)g(2) + f(k). \end{aligned}$$

Since both $f(1)$ and $f(2)$ are odd and either $(1-k)$ or $(2-k)$ is even, $f(k)$ is odd. Thus $f(k) \neq 0 \ \forall\ k \in \mathbb{Z}$.

5.64 Since $\sqrt{\dfrac{3n-5}{n+1}} = \sqrt{3 - \dfrac{8}{n+1}}$ is an integer,

$$\frac{8}{n+1} \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$$

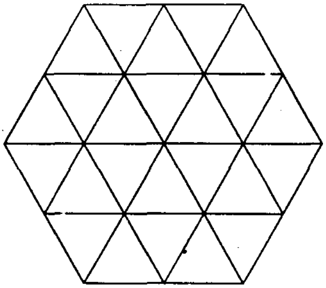
and $3 - \dfrac{8}{n+1}$ is a square. Consequently $\dfrac{8}{n+1} = -1$ or 2; i.e., $n = -9$ or 3.

5.65 We can write $2^{11} + 2^8 + 2^n$ as $2^n(2^{8-n}9 + 1)$. Note that for any $k < 8$, $2^k(2^{8-k}9 + 1)$ is not a square because when k is odd, 2^k is not a square and in the other case, the second factor is not a square. Hence $n \geq 8$. Now write $2^{11} + 2^8 + 2^n$ as $2^8(9 + 2^{n-8})$. Then the problem is to find the number of non-negative integers k such that $9 + 2^k$ is a square. $9 + 2^k = t^2 \Rightarrow 2^k = (t-3)(t+3) \Rightarrow t - 3 = 2^p$ and $t + 3 = 2^{p+q}$ for some non-negative integers p and q. Therefore $2^p(2^q - 1) = 6$ implying $p = 1$ from which it follows that $t = 5$. Hence there is a unique solution.

❻

Combinatorics

6.01 As shown in the figure, partition the hexagon into 24



equilateral triangles. Now by the pigeon-hole principle, two points lie within a triangle and this pair of points serves our purpose.

6.02 Let

$$T = \{24k + \ell : 0 \leq k \leq 4 \text{ and } 1 \leq \ell \leq 12\} \cap S.$$

Then $|T| = 52$ and difference of any two distinct elements in T is different from 12. Let B be the set got from A by replacing any element in A, of the form $12k + \ell$ where k is odd and $1 \leq \ell \leq 12$, by $12(k-1) + \ell$ and leaving other elements, as they are. Then $|A| = |B|$ and $B \subseteq T$. Therefore $|A| \leq 52$.

Second Solution Define a graph G as follows: The vertex set of G is S; two vertices x, y are joined if $|x - y| = 12$. Then G is a union of 4 paths of order 9 and 8 paths of order 8. Now for any $T \subseteq S$, T is independent (i.e., no two vertices of T are joined) $\iff$ for all $x, y \in T$, $|x - y| \neq$

12. Cardinality of any maximum independent set in a path of order 9 (resp. 8) is 5 (resp. 4). Therefore a maximum independent set in G has $5 \times 4 + 4 \times 8 = 52$ elements.

6.03 For $k = 1, 2, \ldots, 11$, let

$$A_k = \{n \in A : n \equiv k \pmod{11}\}.$$

Then $|A_1| = 10$ and for $k = 2, 3, \ldots, 11$, $|A_k| = 9$. The set

$$\{k : 1 \leq k \leq 10 \text{ and } B \cap A_k \neq \emptyset\}$$

has at most 5 elements for otherwise therefore would be two numbers ℓ, m in this set such that $\ell + m = 11$ implying $11 \mid x + y$ where $x \in B \cap A_\ell$ and $y \in B \cap A_m$. This together with $|B \cap A_{11}| \leq 1$ implies

$$|B| = \sum_{k=1}^{11} |B \cap A_k| \leq 10 + 4 \cdot 9 + 1 = 47.$$

6.04 Place twelve 1s in a row:

1 1 1 1 1 1 1 1 1 1 1 1.

For any placement of two + signs in the 11 positions available in between these ones—both signs should not occupy the same position—let us associate a triple of positive integers (x, y, z) where x is the number of ones preceding the first sign, y is the number of ones in between the signs and z is the number of ones after the second sign; $(x, y, z) \in A$ and this association is bijective. Therefore $|A| = \binom{11}{2} = 55$.

6.05 For $i = 1, 2, \ldots, 6$. Let A_i (resp. B_i) be the set of days on which ith friend is present (resp. absent) at dinner. Given that

$$\begin{aligned} |A_i| = |B_i| &= 7 \\ |A_i \cap A_j| &= 5 \\ |A_i \cap A_j \cap A_k| &= 4 \\ |A_i \cap A_j \cap A_k \cap A_\ell| &= 3 \\ |A_i \cap A_j \cap A_k \cap A_\ell \cap A_m| &= 2 \\ \text{and} \quad |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6| &= 1 \end{aligned}$$

where i, j, k, ℓ, m vary from 1 to 6 and are distinct. Number of dinners at which at least one friend was present

$$\begin{aligned} &= |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6| \\ &= \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \\ &\quad \sum |A_i \cap A_j \cap A_k \cap A_\ell| + \\ &\quad \sum |A_i \cap A_j \cap A_k \cap A_\ell \cap A_m| - \\ &\quad |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6| \\ &= \binom{6}{1}7 - \binom{6}{2}5 + \binom{6}{3}4 - \binom{6}{4}3 + \binom{6}{5}2 - \binom{6}{6}1 \\ &= 13. \end{aligned}$$

The total number of dinners is $|A_1| + |B_1| = 14$. Hence the number of dinners I had alone is $14 - 13 = 1$.

6.06 Let us prove the following generalization: Let A be a set of objects such that weight of each object is a positive integer and the total weight of all objects is an even integer not more than $2\,|A|$. If $|A|$ is even and none of the weights exceeds $|A|$, then A can be partitioned into subsets X, Y such that

$$\sum_{x \in X} w(x) = \sum_{x \in Y} w(x) \qquad \cdots (1)$$

where $w(x)$ denotes the weight of the object x.

Let $B = \{x_1, \ldots, x_m\}$ be the subset of all elements in A of weight ≥ 2 and $C = A \setminus B$; let $|C| = n$. Since the total of the weights $\sum_{i=1}^{m} w(x_i) + n$ is even and not greater than $2(m+n)$, we can write

$$\sum_{i=1}^{m} w(x_i) + n + 2\alpha = 2(m+n)$$

where α is a non-negative integer. Then

$$\sum_{i=1}^{m} (w(x_i) - 2) + 2\alpha = n. \qquad \cdots (2)$$

Case 1 m is even.

Let $m = 2k$. Now any partition X, Y of A such that for $i = 1, \ldots, k$, $x_i \in X$, for $x = k+1, \ldots, 2k$, $x_i \in Y$, $\sum_{i=k+1}^{2k} (w(x_i) - 2) + \alpha$ elements of C are in X and $\sum_{i=1}^{k} (w(x_i) - 2) + \alpha$ elements of C are in Y, serves our purpose.

Case 2 m is odd.

Let $m = 2k - 1$. Since $2k - 1 + n = |A|$ is even, n is odd. Therefore $\sum_{i=1}^{m} (w(x_i) - 2)$ being odd by (2), is positive. Therefore for some $i \leq m$, $w(x_i) - 2 > 0$; assume that $w(x_1) > 2$. Let X, Y be a partition of A such that $x_1, \ldots, x_{k-1} \in X$, $x_k, x_{k+1}, \ldots, x_{2k-1} \in Y$, $\sum_{i=k}^{2k-1} (w(x_i) - 2) + \alpha + 1$ elements of C are in X and $\sum_{i \leq (k-1)} (w(x_i) - 2) + \alpha - 1$ elements of C are in Y. [Note that $\sum_{i\leq k-1}(w(x_i)-2)+\alpha-1$ is non-negative, for otherwise $\alpha=0$ and $k-1=0$ ($\because\ w(x_1)-2>0$) implying by (2) $w(x_1)=2+n=|A|+1$, a contradiction.] It is easy to see that (1) holds.

6.07 Let x (resp. y) be the number of days on which 9 (resp. 10) soldiers do guard duty. Let k be the number of days on which every soldier gets duty. Then

$$9x+10y=33k. \qquad \cdots (1)$$

So we have to minimize $x+y$ subject to (1). Obviously when $k=1$, (1) has no non-negative integral solution for x,y. So $k\geq 2$, then $10(x+y)\geq 9x+10y=33k\geq 66 \Rightarrow x+y\geq 7$. $x+y$ attains this minimum when $k=2$; in that case, $y=3$, and $x=4$.

Here is one way of allocating the duty to the soldiers.

Let $D_1=\{1,2,3,4,5,6,7,8,9\}$,
$D_2=\{10,11,12,13,14,15,16,17,18\}$,
$D_3=\{19,20,21,22,23,24,25,26,27\}$,
$D_4=\{28,29,30,31,32,33,1,2,3\}$,
$D_5=\{4,5,6,7,8,9,10,11,12,13\}$,
$D_6=\{14,15,16,17,18,19,20,21,22,23\}$
and $D_7=\{24,25,26,27,28,29,30,31,32,33\}$.

For $k=1,\ldots,7$, and $j=1,\ldots,33$, the soldier S_j is assigned duty on k^{th} day, if $j\in D_k$.

6.08 Since the books are identical and the bags are indistinguishable, each way of distributing the books is given by specifying the number of books in various bags. In the first case, 100 books are to be distributed. So each way of doing this is determined by a finite sequence of positive integers $a_1, a_2, \ldots, a_{10}$ such that $a_1 < a_2 < \cdots < a_{10}$ and $a_1 + a_2 + \cdots + a_{10} = 100$. Each way of distributing 55 books in the second case is given by a finite sequence of positive integers $b_1, b_2, \ldots, b_{10}$ such that $b_1 \leq b_2 \leq \cdots \leq b_{10}$ and $b_1 + \cdots + b_{10} = 55$. Therefore we have to show that the sets

$$\mathcal{A} = \left\{ (a_1, \ldots a_{10}) \;\middle|\; \begin{array}{l} a_i \in \mathbb{N} \text{ for } i = 1, \ldots 10, \\ a_1 < a_2 < \cdots < a_{10} \\ \text{and } a_1 + a_2 \cdots + a_{10} = 100 \end{array} \right\}$$

$$\text{and } \mathcal{B} = \left\{ (b_1, \ldots b_{10}) \;\middle|\; \begin{array}{l} b_i \in \mathbb{N} \text{ for } i = 1, \ldots 10, \\ b_1 \leq b_2 \leq \cdots \leq b_{10} \\ \text{and } b_1 + b_2 \cdots + b_{10} = 55 \end{array} \right\}$$

have the same cardinality. For any $(a_1, \ldots, a_{10}) \in \mathcal{A}$ associate a sequence $(b_1, \ldots, b_{10})$ where $b_i = a_i - i + 1$, for $i = 1, \ldots, 10$. It is easy to verify that $(b_1, \ldots, b_{10}) \in \mathcal{B}$ and this association is a bijection from $\mathcal{A}$ to $\mathcal{B}$.

6.09 Let us make use of the following facts about arithmetic progressions:

(1) The min/max of an A.P. occurs at one of the end terms.

(2) The difference of the end terms of an A.P. having n terms, is divisible by $n - 1$.

Let a_{ij} denote the number in the i^{th} row and j^{th} column. By (1), at one of the corners 1 occurs. Let us assume $a_{11} = 1$. By (2) we have

(3) $a_{1n} \equiv a_{nn} \equiv a_{n1} \equiv 1 \pmod{n-1}$.

By (1) and (3), 2 cannot appear anywhere in the submatrix got by deleting 1st row and 1st column; obviously it should be adjacent to 1; i.e., either a_{12} or a_{21} is 2; assume that $a_{12} = 2$; then $a_{1j} = j$, for $j = 1, \ldots, n$.

Let us find where $n + 1$ is placed. Again by (1) it is one of the corners of the submatrix got by deleting the first row; by (3) it can be neither a_{n1} nor a_{nn}; it could not be a_{2n} for otherwise $a_{nn} = 2n - 1 \Rightarrow a_{n1} = n^2$ by (1) $\Rightarrow$ common-difference of the A.P. in the first column $= n+1 \Rightarrow a_{21} = n+2$, an impossibility. $\therefore\ a_{21} = n+1$. Now the terms of the first column are known; $a_{n1} = 1+(n-1)n = n^2-n+1$; by (1), $a_{nn} = n^2$; therefore the terms of the last row are found; thus the entire arrangement is determined. It is given by:

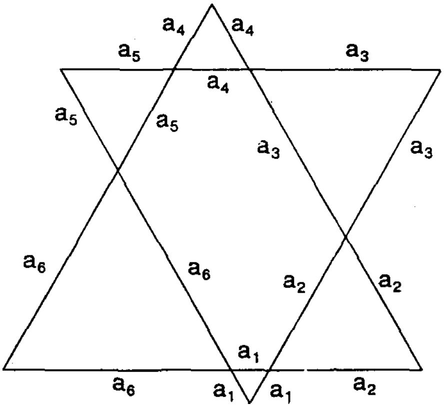
$$a_{ij} = (i-1)n + j, \ 1 \leq i, j \leq n.$$

Thus the arrangement is determined by the positioning of 1 and 2; 1 can occupy one the four corners and 2 can be at one of the two adjacent squares of 1. Therefore the number of arrangements is $4 \times 2 = 8$.

6.10 Let $a_1, \ldots, a_6$ be the sides of the polygon. From the figure, we have the following relations:

$$\begin{aligned} &(1) \quad a_1 + a_2 = a_4 + a_5 \\ &(2) \quad a_2 + a_3 = a_5 + a_6 \\ &(3) \quad a_1 + a_6 = a_3 + a_4. \end{aligned}$$

Let us assume $a_1 = 1$. Then $a_4 + a_5 \geq 5 \Rightarrow a_2 \geq 4$ by (1). Similarly $a_6 \geq 4$. Therefore $a_2, a_6 \in \{4, 5, 6\}$. Thus one of the sides adjacent to a_1 is either 4 or 5; so up to rotational symmetry, we can assume $a_2 = 4$ or 5.



Let $a_2 = 4$. Then by (1) we have $a_4 + a_5 = 5 \Rightarrow \{a_4, a_5\} = \{2, 3\}$. $\therefore$ $\{a_3, a_6\} = \{5, 6\}$; (3) $\Rightarrow 1 + a_6 = a_3 + a_4 \geq 5 + 2 \Rightarrow a_6 = 6$; therefore $a_3 = 5$, $a_4 = 2$ and $a_5 = 3$.

Now let $a_2 = 5$. Then by (1) we have $a_4 + a_5 = 6 \Rightarrow \{a_4, a_5\} = \{2, 4\}$. $\therefore$ $\{a_3, a_6\} = \{3, 6\} \Rightarrow a_6 = 6$ ($\because$ $a_6 \geq 4$ by (3)). $\therefore$ $a_3 = 3$. Now (2) $\Rightarrow a_5 = 2$. Therefore $a_4 = 4$.

Hence $(a_1, a_2, \ldots, a_6)$ is a cyclic rotation of either $(1, 4, 5, 2, 3, 6)$ or $(1, 5, 3, 4, 2, 6)$.

6.11 Let the strength of the class be 100. Let T, E, M, S be the sets of students who passed in Tamil, English, Mathematics and Science, respectively. Then the number of students who failed in all subjects is

$$\begin{aligned} 100 - |T \cup E \cup M \cup S| &\geq 100 - (|T| + |E| + |M| + |S|) \\ &= 100 - (30 + 25 + 20 + 15) = 10. \end{aligned}$$

Note that equality holds when

$$|T \cup E \cup M \cup S| = |T| + |E| + |M| + |S|;$$

i.e., when these sets are pairwise disjoint.

6.12 There is *no way* of putting nine letters in the right envelopes and the tenth wrongly.

6.13 Let these classes be respectively $\mathcal{A}$ and $\mathcal{B}$. For any $A = \{a, b, c\} \in \mathcal{A}$, associate $B = \{64-a, 64-b, 64-c\}$. Since $(64-a)+(64-b)+(64-c) = 192-(a+b+c) > 97$, $B \in \mathcal{B}$. This association is obviously one-to-one. Further there are elements in $\mathcal{B}$ which are associated with no element of $\mathcal{A}$—for example $(31, 32, 33)$. Therefore $|\mathcal{A}| < |\mathcal{B}|$.

GEMS FROM
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